Count Panel Data

Example Questions and Solutions

230347: Advanced Microeconometrics

Question 1

Conditional Likelihood for Static Fixed Effects Poisson

The likelihood for individual i in a static fixed effects Poisson model is given by:

$$\begin{aligned}\Pr\left(y_{i1},\ldots,y_{iT}\right) &= \prod_{t=1}^{T}\frac{\exp\left(-\alpha_i\lambda_{it}\right)\left(\alpha_i\lambda_{it}\right)^{y_{it}}}{y_{it}!}\\ &= \frac{\exp\left(-\alpha_i\sum_{t=1}^{T}\lambda_{it}\right)\prod_{t=1}^{T}\alpha_i^{y_{it}}\prod_{t=1}^{T}\lambda_{it}^{y_{it}}}{\prod_{t=1}^{T}y_{it}!}\end{aligned}$$

Show that the likelihood conditional on $\sum_{t=1}^{T}y_{it}$ does not depend on the fixed effect α_i. For this, you should use the following theorem about Poisson-distributed random variables:

> If $Y_t \sim \mathcal{P}\left(\mu_t\right)$, $t = 1,2,\ldots,T$, are independent random variables and if $\sum_{t=1}^{T}\mu_t < \infty$, then $S_Y = \sum_{t=1}^{T}Y_t \sim \mathcal{P}\left(\sum_{t=1}^{T}\mu_t\right)$

Solution

Using the fact about the sum of independent Poisson-distributed random variables:

$$\begin{aligned}\Pr\left(\sum_{t=1}^{T}y_{it}\right) &= \frac{\exp\left(-\sum_{t=1}^{T}\alpha_i\lambda_{it}\right)\left(\sum_{t=1}^{T}\alpha_i\lambda_{it}\right)^{\sum_{t=1}^{T}y_{it}}}{\left(\sum_{t=1}^{T}y_{it}\right)!}\\ &= \frac{\exp\left(-\alpha_i\sum_{t=1}^{T}\lambda_{it}\right)\prod_{t=1}^{T}\left(\alpha_i\sum_{s=1}^{T}\lambda_{is}\right)^{y_{it}}}{\left(\sum_{t=1}^{T}y_{it}\right)!}\\ &= \frac{\exp\left(-\alpha_i\sum_{t=1}^{T}\lambda_{it}\right)\prod_{t=1}^{T}\alpha_i^{y_{it}}\left(\sum_{s=1}^{T}\lambda_{is}\right)^{y_{it}}}{\left(\sum_{t=1}^{T}y_{it}\right)!}\end{aligned}$$

The conditional likelihood is then:

$$
\begin{aligned}
\Pr\left(y_{i1},\dots,y_{iT}\left|\sum_{t=1}^{T}y_{it}\right.\right) &= \frac{\Pr\left(y_{i1},\dots,y_{iT},\sum_{t=1}^{T}y_{it}\right)}{\Pr\left(\sum_{t=1}^{T}y_{it}\right)} \\
&= \frac{\Pr\left(y_{i1},\dots,y_{iT}\right)}{\Pr\left(\sum_{t=1}^{T}y_{it}\right)} \\
&= \frac{\frac{\exp\left(-\alpha_i\sum_{t=1}^{T}\lambda_{it}\right)\prod_{t=1}^{T}\alpha_i^{y_{it}}\prod_{t=1}^{T}\lambda_{it}^{y_{it}}}{\prod_{t=1}^{T}y_{it}!}}{\frac{\exp\left(-\alpha_i\sum_{t=1}^{T}\lambda_{it}\right)\prod_{t=1}^{T}\alpha_i^{y_{it}}\left(\sum_{s=1}^{T}\lambda_{is}\right)^{y_{it}}}{\left(\sum_{t=1}^{T}y_{it}\right)!}} \\
&= \frac{\left(\sum_{t=1}^{T}y_{it}\right)!}{\prod_{t=1}^{T}y_{it}!}\times\frac{\prod_{t=1}^{T}\lambda_{it}^{y_{it}}}{\prod_{t=1}^{T}\left(\sum_{s=1}^{T}\lambda_{is}\right)^{y_{it}}}
\end{aligned}
$$

which does not depend on α_i.

Question 2

Linear Feedback Model

Consider the model:

$$
y_{it} = \rho y_{it-1} + \alpha_i \exp\left(\beta x_{it}\right) + u_{it} \qquad i = 1,\dots,N \quad \text{and } t = 1,2,3
$$

where $\mathbb{E}\left[u_{it}\right|x_{i1},\dots,x_{it},y_{i1},\dots,y_{it-2}] = 0$. Notice that x_{it} is a scalar, and that $T = 3$.

(i) Show that for the third time period for individual i there is a quasi-differencing transformation $q_{i3}\left(\boldsymbol{\theta}\right)$, where $\boldsymbol{\theta} = (\rho,\beta)'$, that removes the fixed effect α_i and satisfies $\mathbb{E}\left[q_{i3}\left(\boldsymbol{\theta}\right)|x_{i1},x_{i2},x_{i3},y_{i1}\right] = 0$.

(ii) Write down a valid instrument matrix $\boldsymbol{Z}_i$ using as many instruments that are available and show that $\mathbb{E}\left[\boldsymbol{Z}_i' q_{i3}\left(\boldsymbol{\theta}\right)\right] = \mathbf{0}$.

(iii) Describe how you would estimate $\boldsymbol{\theta}$ with one-step GMM to get the first-step estimates $\widehat{\boldsymbol{\theta}}_1$. You do not need to specify the first-step weight matrix.

(iv) The estimator of the variance-covariance matrix for the second-step GMM estimates $\widehat{\boldsymbol{\theta}}_2$ is:

$$
\widehat{\mathrm{Var}}\left(\widehat{\boldsymbol{\theta}}_2\right) = \frac{1}{N}\left[\left(\frac{1}{N}\sum_{i=1}^{N}\left.\frac{\partial \boldsymbol{m}_i\left(\boldsymbol{\theta}\right)}{\partial\boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}_2}\right)'\boldsymbol{W}_2\left(\widehat{\boldsymbol{\theta}}_1\right)^{-1}\left(\frac{1}{N}\sum_{i=1}^{N}\left.\frac{\partial \boldsymbol{m}_i\left(\boldsymbol{\theta}\right)}{\partial\boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}_2}\right)\right]^{-1}
$$

where $\boldsymbol{m}_i\left(\boldsymbol{\theta}\right) = \boldsymbol{Z}_i' q_{i3}\left(\boldsymbol{\theta}\right)$ and where

$$
\boldsymbol{W}_2\left(\widehat{\boldsymbol{\theta}}_1\right) = \frac{1}{N}\sum_{i=1}^{N}\left[\boldsymbol{Z}_i' q_{i3}\left(\widehat{\boldsymbol{\theta}}_1\right)\right]\left[\boldsymbol{Z}_i' q_{i3}\left(\widehat{\boldsymbol{\theta}}_1\right)\right]'
$$

is the second-step weight matrix. Write down $\frac{1}{N}\sum_{i=1}^{N}\left.\frac{\partial \boldsymbol{m}_i(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}_1}$ explicitly.

Solution

(i) Let $\lambda_{it} = \exp(\beta x_{it})$. Then:

$$\begin{aligned} q_{i3}(\boldsymbol{\theta}) &= (y_{i3} - \rho y_{i2}) \frac{\lambda_{i2}}{\lambda_{i3}} - (y_{i2} - \rho y_{i1}) \\ &= (\alpha_i \lambda_{i3} + u_{i3}) \frac{\lambda_{i2}}{\lambda_{i3}} - (\alpha_i \lambda_{i2} + u_{i2}) \\ &= \alpha_i \lambda_{i2} + u_{i3} \frac{\lambda_{i2}}{\lambda_{i3}} - \alpha_i \lambda_{i2} - u_{i2} \\ &= u_{i3} \frac{\lambda_{i2}}{\lambda_{i3}} - u_{i2} \end{aligned}$$

So:

$$\begin{aligned} \mathbb{E}\left[q_{i3}(\boldsymbol{\theta}) \,|x_{i1}, x_{i2}, x_{i3}, y_{i1}\right] &= \mathbb{E}\left[u_{i3} \frac{\lambda_{i2}}{\lambda_{i3}} - u_{i2} \middle| x_{i1}, x_{i2}, x_{i3}, y_{i1}\right] \\ &= \frac{\lambda_{i2}}{\lambda_{i3}} \mathbb{E}\left[u_{i3} | x_{i1}, x_{i2}, x_{i3}, y_{i1}\right] - \mathbb{E}\left[u_{i2} | x_{i1}, x_{i2}, x_{i3}, y_{i1}\right] \\ &= 0 \end{aligned}$$

(ii) Since $T = 3$ we can only use the last time period (since we need two lags of y_{it} for the quasi-differencing). A valid instrument matrix using all available instruments is:

$$\boldsymbol{Z}_i = \begin{pmatrix} x_{i1} & x_{i2} & x_{i3} & y_{i1} \end{pmatrix}$$

So

$$\boldsymbol{Z}_i' q_{i3}(\boldsymbol{\theta}) = \begin{pmatrix} x_{i1} q_{i3}(\boldsymbol{\theta}) \\ x_{i2} q_{i3}(\boldsymbol{\theta}) \\ x_{i3} q_{i3}(\boldsymbol{\theta}) \\ y_{i1} q_{i3}(\boldsymbol{\theta}) \end{pmatrix}$$

By the law of iterated expectations:

$$\mathbb{E}\left[\boldsymbol{Z}_i' q_{i3}(\boldsymbol{\theta})\right] = \mathbb{E}\left[\mathbb{E}\left[\boldsymbol{Z}_i' q_{i3}(\boldsymbol{\theta}) \middle| \boldsymbol{Z}_i\right]\right] = \mathbb{E}\left[\boldsymbol{Z}_i' \underbrace{\mathbb{E}\left[q_{i3}(\boldsymbol{\theta}) | \boldsymbol{Z}_i\right]}_{=0}\right] = \boldsymbol{0}$$

(iii) The sample analogue of our moment condition $\mathbb{E}\left[\boldsymbol{Z}_i' q_{i3}(\boldsymbol{\theta})\right] = \boldsymbol{0}$ is $\frac{1}{N}\sum_{i=1}^N \boldsymbol{m}_i(\boldsymbol{\theta}) = \frac{1}{N}\sum_{i=1}^N \boldsymbol{Z}_i' q_{i3}(\boldsymbol{\theta})$, or more explicitly:

$$\frac{1}{N}\sum_{i=1}^N \boldsymbol{m}_i(\boldsymbol{\theta}) = \begin{pmatrix} \frac{1}{N}\sum_{i=1}^N x_{i1} \left[(y_{i3} - \rho y_{i2}) \frac{\lambda_{i2}}{\lambda_{i3}} - (y_{i2} - \rho y_{i1})\right] \\ \frac{1}{N}\sum_{i=1}^N x_{i2} \left[(y_{i3} - \rho y_{i2}) \frac{\lambda_{i2}}{\lambda_{i3}} - (y_{i2} - \rho y_{i1})\right] \\ \frac{1}{N}\sum_{i=1}^N x_{i3} \left[(y_{i3} - \rho y_{i2}) \frac{\lambda_{i2}}{\lambda_{i3}} - (y_{i2} - \rho y_{i1})\right] \\ \frac{1}{N}\sum_{i=1}^N y_{i1} \left[(y_{i3} - \rho y_{i2}) \frac{\lambda_{i2}}{\lambda_{i3}} - (y_{i2} - \rho y_{i1})\right] \end{pmatrix}$$

The first-step estimate is then:

$$\widehat{\boldsymbol{\theta}}_1 = \arg\min_{\boldsymbol{\theta}} \left[\frac{1}{N}\sum_{i=1}^{N} \boldsymbol{m}_i(\boldsymbol{\theta})\right]' \boldsymbol{W}_1^{-1} \left[\frac{1}{N}\sum_{i=1}^{N} \boldsymbol{m}_i(\boldsymbol{\theta})\right]$$

where $\boldsymbol{W}_1$ is a 3×3 positive definite matrix.

(iv) Recall

$$\begin{aligned} q_{i3}(\boldsymbol{\theta}) &= (y_{i3} - \rho y_{i2}) \frac{\lambda_{i2}}{\lambda_{i3}} - (y_{i2} - \rho y_{i1}) \\ &= (y_{i3} - \rho y_{i2}) \exp(\beta (x_{i2} - x_{i3})) - (y_{i2} - \rho y_{i1}) \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial q_{i3}(\boldsymbol{\theta})}{\partial \rho} &= -y_{i2} \frac{\lambda_{i2}}{\lambda_{i3}} + y_{i1} \\ \frac{\partial q_{i3}(\boldsymbol{\theta})}{\partial \beta} &= (y_{i3} - \rho y_{i2}) \frac{\lambda_{i2}}{\lambda_{i3}} (x_{i2} - x_{i3}) \end{aligned}$$

The full derivative matrix is then:

$$\frac{1}{N}\sum_{i=1}^{N} \frac{\partial \boldsymbol{m}_i(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \begin{pmatrix} \frac{1}{N}\sum_{i=1}^{N} x_{i1} \left[-y_{i2}\frac{\lambda_{i2}}{\lambda_{i3}} + y_{i1}\right] & \frac{1}{N}\sum_{i=1}^{N} x_{i1} \left[(y_{i3} - \rho y_{i2}) \frac{\lambda_{i2}}{\lambda_{i3}} (x_{i2} - x_{i3})\right] \\ \frac{1}{N}\sum_{i=1}^{N} x_{i2} \left[-y_{i2}\frac{\lambda_{i2}}{\lambda_{i3}} + y_{i1}\right] & \frac{1}{N}\sum_{i=1}^{N} x_{i2} \left[(y_{i3} - \rho y_{i2}) \frac{\lambda_{i2}}{\lambda_{i3}} (x_{i2} - x_{i3})\right] \\ \frac{1}{N}\sum_{i=1}^{N} x_{i3} \left[-y_{i2}\frac{\lambda_{i2}}{\lambda_{i3}} + y_{i1}\right] & \frac{1}{N}\sum_{i=1}^{N} x_{i3} \left[(y_{i3} - \rho y_{i2}) \frac{\lambda_{i2}}{\lambda_{i3}} (x_{i2} - x_{i3})\right] \\ \frac{1}{N}\sum_{i=1}^{N} y_{i1} \left[-y_{i2}\frac{\lambda_{i2}}{\lambda_{i3}} + y_{i1}\right] & \frac{1}{N}\sum_{i=1}^{N} y_{i1} \left[(y_{i3} - \rho y_{i2}) \frac{\lambda_{i2}}{\lambda_{i3}} (x_{i2} - x_{i3})\right] \end{pmatrix}$$

Evaluating this at $\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}_2$ gives the desired answer.

Question 3

Static Poisson with Gamma-Distributed Random Effects

The model is $y_{it}|\alpha_i, \lambda_{it} \overset{iid}{\sim} Poisson(\alpha_i \lambda_{it})$ where $\lambda_{it} = \exp(\boldsymbol{x}_{it}'\boldsymbol{\beta})$. Given the Poisson distribution:

$$\Pr(y_{it}|\lambda_{it}, \alpha_i) = \frac{\exp(-\alpha_i \lambda_{it}) (\alpha_i \lambda_{it})^{y_{it}}}{y_{it}!}$$

α_i is distributed according to a Gamma distribution with shape and rate $\delta > 0$, so the density of α_i is $f(\alpha_i|\delta) = \frac{\delta^\delta}{\Gamma(\delta)} \alpha_i^{\delta - 1} \exp(-\alpha_i \delta)$. $\Gamma(x)$ is the Gamma function defined by $\Gamma(x) = \int_0^\infty s^{x-1} e^{-s} ds$.

(i) Show that the joint density $\Pr(y_{i1}, \ldots, y_{iT})$ can be written as:

$$\Pr(y_{i1}, \ldots, y_{iT}|\boldsymbol{\lambda}_i, \alpha_i) = \prod_{t=1}^{T} \frac{\lambda_{it}^{y_{it}}}{y_{it}!} \left[\exp\left(-\alpha_i \sum_{t=1}^{T} \lambda_{it}\right) \times \alpha_i^{\sum_{t=1}^{T} y_{it}}\right]$$

where $\boldsymbol{\lambda}_i = (\lambda_{i1}, \ldots, \lambda_{iT})$.

(ii) Show that:

$$\int_0^\infty \Pr(y_{i1},\ldots,y_{iT}|\boldsymbol{\lambda}_i,\alpha_i)\, f(\alpha_i|\delta)\, d\alpha_i = \left[\prod_{t=1}^{T}\frac{\lambda_{it}^{y_{it}}}{y_{it}!}\right]\frac{\delta^\delta}{\Gamma(\delta)}\int_0^\infty \exp\left(-\alpha_i\left(\sum_{t=1}^{T}\lambda_{it}+\delta\right)\right)\alpha_i^{\sum_{t=1}^{T}y_{it}+\delta-1}d\alpha_i$$

(iii) Using (ii) and a property[1] of the Gamma distribution, $\int_0^\infty v^{x-1}e^{-bv}dv = b^{-x}\Gamma(x)$, show that we can integrate out the random effect:

$$\begin{aligned}\Pr(y_{i1},\ldots,y_{iT}|\boldsymbol{\lambda}_i,\delta) &= \int_0^\infty \Pr(y_{i1},\ldots,y_{iT}|\boldsymbol{\lambda}_i,\alpha_i)\, f(\alpha_i|\delta)\, d\alpha_i \\ &= \left[\prod_{t=1}^{T}\frac{\lambda_{it}^{y_{it}}}{y_{it}!}\right]\frac{\Gamma\left(\sum_{t=1}^{T}y_{it}+\delta\right)}{\Gamma(\delta)}\left(\frac{\delta}{\sum_{t=1}^{T}\lambda_{it}+\delta}\right)^{\delta}\left(\sum_{t=1}^{T}\lambda_{it}+\delta\right)^{-\sum_{t=1}^{T}y_{it}}\end{aligned}$$

Solution

(i)

$$\begin{aligned}\Pr(y_{i1},\ldots,y_{iT}|\boldsymbol{\lambda}_i,\alpha_i) &= \prod_{t=1}^{T}\Pr(y_{it}|\lambda_{it},\alpha_i) \\ &= \prod_{t=1}^{T}\frac{\exp(-\alpha_i\lambda_{it})(\alpha_i\lambda_{it})^{y_{it}}}{y_{it}!} \\ &= \prod_{t=1}^{T}\frac{\lambda_{it}^{y_{it}}}{y_{it}!}\left[\exp\left(-\alpha_i\sum_{t=1}^{T}\lambda_{it}\right)\times\alpha_i^{\sum_{t=1}^{T}y_{it}}\right]\end{aligned}$$

(ii) Substituting this into the joint likelihood and rearranging terms:

$$\begin{aligned}\int_0^\infty \Pr(y_{i1},\ldots,y_{iT}|\boldsymbol{\lambda}_i,\alpha_i)\, f(\alpha_i|\delta)\, d\alpha_i &= \int_0^\infty\prod_{t=1}^{T}\frac{\lambda_{it}^{y_{it}}}{y_{it}!}\left[\exp\left(-\alpha_i\sum_{t=1}^{T}\lambda_{it}\right)\times\alpha_i^{\sum_{t=1}^{T}y_{it}}\right]f(\alpha_i|\delta)\,d\alpha_i \\ &= \left[\prod_{t=1}^{T}\frac{\lambda_{it}^{y_{it}}}{y_{it}!}\right]\int_0^\infty\left[\exp\left(-\alpha_i\sum_{t=1}^{T}\lambda_{it}\right)\times\alpha_i^{\sum_{t=1}^{T}y_{it}}\times\frac{\delta^\delta}{\Gamma(\delta)}\times\alpha_i^{\delta-1}\exp(-\alpha_i\delta)\right]d\alpha_i \\ &= \left[\prod_{t=1}^{T}\frac{\lambda_{it}^{y_{it}}}{y_{it}!}\right]\frac{\delta^\delta}{\Gamma(\delta)}\int_0^\infty\left[\left(\exp\left(-\alpha_i\sum_{t=1}^{T}\lambda_{it}\right)\alpha_i^{\sum_{t=1}^{T}y_{it}}\right)\alpha_i^{\delta-1}\exp(-\alpha_i\delta)\right]d\alpha_i \\ &= \left[\prod_{t=1}^{T}\frac{\lambda_{it}^{y_{it}}}{y_{it}!}\right]\frac{\delta^\delta}{\Gamma(\delta)}\int_0^\infty\exp\left(-\alpha_i\left(\sum_{t=1}^{T}\lambda_{it}+\delta\right)\right)\alpha_i^{\sum_{t=1}^{T}y_{it}+\delta-1}d\alpha_i\end{aligned}$$

(iii) Using the given property of the Gamma distribution for the integral term in our likelihood, where $v=\alpha_i$, $b=\sum_{t=1}^{T}\lambda_{it}+\delta$ and $x=\sum_{t=1}^{T}y_{it}+\delta$:

$$\int_0^\infty\exp\left(-\alpha_i\left(\sum_{t=1}^{T}\lambda_{it}+\delta\right)\right)\alpha_i^{\sum_{t=1}^{T}y_{it}+\delta-1}d\alpha_i = \left(\sum_{t=1}^{T}\lambda_{it}+\delta\right)^{-\sum_{t=1}^{T}y_{it}-\delta}\Gamma\left(\sum_{t=1}^{T}y_{it}+\delta\right)$$

[1] This comes from, where $s=bv$,

$$\Gamma(x) = \int_0^\infty s^{x-1}e^{-s}ds = \int_0^\infty (bv)^{x-1}e^{-bv}b\,dv = b^x\int_0^\infty v^{x-1}e^{-bv}dv$$

Substituting this back into the likelihood (using $\Pr\left(y_{i1},\ldots,y_{iT}|\boldsymbol{\lambda}_i,\delta\right) = \int_0^\infty \Pr\left(y_{i1},\ldots,y_{iT}|\boldsymbol{\lambda}_i,\alpha_i\right) f\left(\alpha_i|\delta\right) d\alpha_i$):

$$\begin{aligned}
\Pr\left(y_{i1},\ldots,y_{iT}|\boldsymbol{\lambda}_i,\delta\right) &= \left[\prod_{t=1}^{T}\frac{\lambda_{it}^{y_{it}}}{y_{it}!}\right]\frac{\delta^\delta}{\Gamma\left(\delta\right)}\left(\sum_{t=1}^{T}\lambda_{it}+\delta\right)^{-\sum_{t=1}^{T}y_{it}-\delta}\Gamma\left(\sum_{t=1}^{T}y_{it}+\delta\right)\\
&= \left[\prod_{t=1}^{T}\frac{\lambda_{it}^{y_{it}}}{y_{it}!}\right]\frac{\Gamma\left(\sum_{t=1}^{T}y_{it}+\delta\right)}{\Gamma\left(\delta\right)}\left(\frac{\delta}{\sum_{t=1}^{T}\lambda_{it}+\delta}\right)^{\delta}\left(\sum_{t=1}^{T}\lambda_{it}+\delta\right)^{-\sum_{t=1}^{T}y_{it}}
\end{aligned}$$