Static Linear Panel Data

Example Questions and Solutions

230347: Advanced Microeconometrics

Question 1

One-Way Error Within Transformation in Matrix Notation

The one-way error component model $y_{it} = \alpha + \boldsymbol{x}'_{it}\boldsymbol{\beta} + \alpha_i + \varepsilon_{it}$ can also be written in matrix form as:

$$\underset{(NT\times 1)}{\boldsymbol{y}} = \underset{(NT\times K)}{\boldsymbol{X}}\underset{(K\times 1)}{\boldsymbol{\beta}} + \underset{(NT\times N)}{(\boldsymbol{I}_N \otimes \boldsymbol{\iota}_T)}\underset{(N\times 1)}{\boldsymbol{\alpha}} + \underset{(N\times T)}{\boldsymbol{\varepsilon}} \tag{1}$$

where:

- $\boldsymbol{y}$ is stacked such that $(y_{11}, \ldots, y_{1T}, y_{21}, \ldots, y_{N1}, \ldots, y_{NT})$.
- $\boldsymbol{X}$ includes a column of ones as its first column.
- $\boldsymbol{I}_k$ is an $k \times k$ identity matrix
- $\boldsymbol{\iota}_k$ is an k vector of ones.
- $\otimes$ is the Kronecker product.

Define the following:

- $\boldsymbol{Z}_\alpha = (\boldsymbol{I}_N \otimes \boldsymbol{\iota}_T)$.
- $\boldsymbol{P} = \boldsymbol{Z}_\alpha \left(\boldsymbol{Z}'_\alpha \boldsymbol{Z}_\alpha\right)^{-1} \boldsymbol{Z}'_\alpha$. The matrix $\boldsymbol{P}$ gets the individual means over time.
- $\boldsymbol{Q} = \boldsymbol{I}_{NT} - \boldsymbol{P}$.

Show that by premultiplying the model in (1) by $\boldsymbol{Q}$ removes the individual effects and that the within estimator for $\boldsymbol{\beta}$ is $\left(\boldsymbol{X}'\boldsymbol{Q}\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\boldsymbol{Q}\boldsymbol{y}$.

Solution

We see that premultiplying the model removes the individual effects:

$$\begin{aligned}
\boldsymbol{y} &= \boldsymbol{X}\boldsymbol{\beta} + (\boldsymbol{I}_N \otimes \boldsymbol{\iota}_T)\,\boldsymbol{\alpha} + \boldsymbol{\varepsilon} \\
\boldsymbol{y} &= \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}_\alpha \boldsymbol{\alpha} + \boldsymbol{\varepsilon} \\
\boldsymbol{Q}\boldsymbol{y} &= \boldsymbol{Q}\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Q}\boldsymbol{Z}_\alpha \boldsymbol{\alpha} + \boldsymbol{Q}\boldsymbol{\varepsilon} \\
\boldsymbol{Q}\boldsymbol{y} &= \boldsymbol{Q}\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{I}_{NT}\boldsymbol{Z}_\alpha \boldsymbol{\alpha} - \boldsymbol{Z}_\alpha \left(\boldsymbol{Z}_\alpha' \boldsymbol{Z}_\alpha\right)^{-1} \boldsymbol{Z}_\alpha' \boldsymbol{Z}_\alpha \boldsymbol{\alpha} + \boldsymbol{Q}\boldsymbol{\varepsilon} \\
\boldsymbol{Q}\boldsymbol{y} &= \boldsymbol{Q}\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}_\alpha \boldsymbol{\alpha} - \boldsymbol{Z}_\alpha \boldsymbol{\alpha} + \boldsymbol{Q}\boldsymbol{\varepsilon} \\
\boldsymbol{Q}\boldsymbol{y} &= \boldsymbol{Q}\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Q}\boldsymbol{\varepsilon}
\end{aligned}$$

The OLS estimator for $\boldsymbol{\beta}$ is then:

$$\begin{aligned}
\widehat{\boldsymbol{\beta}} &= \left((\boldsymbol{Q}\boldsymbol{X})'\,(\boldsymbol{Q})\,\boldsymbol{X}\right)^{-1} (\boldsymbol{Q}\boldsymbol{X})'\,\boldsymbol{Q}\boldsymbol{y} \\
&= \left(\boldsymbol{X}'\boldsymbol{Q}'\boldsymbol{Q}\boldsymbol{X}\right)^{-1} \boldsymbol{X}'\boldsymbol{Q}'\boldsymbol{Q}\boldsymbol{y}
\end{aligned}$$

To simplify this, we show that $\boldsymbol{Q}'\boldsymbol{Q} = \boldsymbol{Q}$, noting that $\boldsymbol{P}$ is symmetric:

$$\begin{aligned}
\boldsymbol{Q}'\boldsymbol{Q} &= (\boldsymbol{I}_{NT} - \boldsymbol{P})'\,(\boldsymbol{I}_{NT} - \boldsymbol{P}) \\
&= \boldsymbol{I}_{NT} - \boldsymbol{P}' - \boldsymbol{P} + \boldsymbol{P}'\boldsymbol{P} \\
&= \boldsymbol{I}_{NT} - 2\boldsymbol{P} + \boldsymbol{Z}_\alpha \left(\boldsymbol{Z}_\alpha' \boldsymbol{Z}_\alpha\right)^{-1} \boldsymbol{Z}_\alpha' \boldsymbol{Z}_\alpha \left(\boldsymbol{Z}_\alpha' \boldsymbol{Z}_\alpha\right)^{-1} \boldsymbol{Z}_\alpha' \\
&= \boldsymbol{I}_{NT} - 2\boldsymbol{P} + \boldsymbol{P} \\
&= \boldsymbol{I}_{NT} - \boldsymbol{P} \\
&= \boldsymbol{Q}
\end{aligned}$$

Therefore the estimator for $\boldsymbol{\beta}$ can be written as $\left(\boldsymbol{X}'\boldsymbol{Q}\boldsymbol{X}\right)^{-1} \boldsymbol{X}'\boldsymbol{Q}\boldsymbol{y}$.

Question 2

Within Transformation and First Differences for $\boldsymbol{T = 2}$

For the model:

$$y_{it} = \boldsymbol{x}_{it}'\boldsymbol{\beta} + \alpha_i + \varepsilon_{it} \qquad i = 1, \ldots, N \quad t = 1, 2$$

Show that for $T = 2$ the within transformation and first differences produce estimates that are numerically identical, i.e. $\widehat{\boldsymbol{\beta}}_{FE} = \widehat{\boldsymbol{\beta}}_{FD}$

Solution

(i) The within estimator is:

$$\begin{aligned}\widehat{\boldsymbol{\beta}}_{FE} &= \left[\sum_{i=1}^{N}\sum_{t=1}^{2}(\boldsymbol{x}_{it}-\bar{\boldsymbol{x}}_i)(\boldsymbol{x}_{it}-\bar{\boldsymbol{x}}_i)'\right]^{-1}\left[\sum_{i=1}^{N}\sum_{t=1}^{2}(\boldsymbol{x}_{it}-\bar{\boldsymbol{x}}_i)(y_{it}-\bar{y}_i)\right] \\ &= \left[\sum_{i=1}^{N}(\boldsymbol{x}_{i1}-\bar{\boldsymbol{x}}_i)(\boldsymbol{x}_{i1}-\bar{\boldsymbol{x}}_i)'+(\boldsymbol{x}_{i2}-\bar{\boldsymbol{x}}_i)(\boldsymbol{x}_{i2}-\bar{\boldsymbol{x}}_i)'\right]^{-1}\left[\sum_{i=1}^{N}(\boldsymbol{x}_{i1}-\bar{\boldsymbol{x}}_i)(y_{i1}-\bar{y}_i)+(\boldsymbol{x}_{i2}-\bar{\boldsymbol{x}}_i)(y_{i2}-\bar{y}_i)\right]\end{aligned}$$

However:

$$\begin{aligned}\boldsymbol{x}_{i1}-\bar{\boldsymbol{x}}_i &= \boldsymbol{x}_{i1}-\frac{\boldsymbol{x}_{i1}+\boldsymbol{x}_{i2}}{2}=\frac{\boldsymbol{x}_{i1}-\boldsymbol{x}_{i2}}{2}=-\frac{\boldsymbol{x}_{i2}-\boldsymbol{x}_{i1}}{2}=-\frac{\Delta\boldsymbol{x}_{i2}}{2} \\ \boldsymbol{x}_{i2}-\bar{\boldsymbol{x}}_i &= \boldsymbol{x}_{i2}-\frac{\boldsymbol{x}_{i1}+\boldsymbol{x}_{i2}}{2}=\frac{\boldsymbol{x}_{i2}-\boldsymbol{x}_{i1}}{2}=\frac{\Delta\boldsymbol{x}_{i2}}{2}\end{aligned}$$

And similarly for $y_{it}-\bar{y}_i$. Using this:

$$\begin{aligned}\widehat{\boldsymbol{\beta}}_{FE} &= \left[\sum_{i=1}^{N}\left(-\frac{\Delta\boldsymbol{x}_{i2}}{2}\right)\left(-\frac{\Delta\boldsymbol{x}_{i2}}{2}\right)'+\left(\frac{\Delta\boldsymbol{x}_{i2}}{2}\right)\left(\frac{\Delta\boldsymbol{x}_{i2}}{2}\right)'\right]^{-1}\left[\sum_{i=1}^{N}\left(-\frac{\Delta\boldsymbol{x}_{i2}}{2}\right)\left(-\frac{\Delta y_{i2}}{2}\right)'+\left(\frac{\Delta\boldsymbol{x}_{i2}}{2}\right)\left(\frac{\Delta y_{i2}}{2}\right)'\right] \\ &= \left[\sum_{i=1}^{N}\frac{1}{4}(\Delta\boldsymbol{x}_{i2})(\Delta\boldsymbol{x}_{i2})'+\frac{1}{4}(\Delta\boldsymbol{x}_{i2})(\Delta\boldsymbol{x}_{i2})'\right]^{-1}\left[\sum_{i=1}^{N}\frac{1}{4}(\Delta\boldsymbol{x}_{i2})(\Delta y_{i2})'+\frac{1}{4}(\Delta\boldsymbol{x}_{i2})(\Delta y_{i2})'\right] \\ &= \left[\sum_{i=1}^{N}(\Delta\boldsymbol{x}_{i2})(\Delta\boldsymbol{x}_{i2})'\right]^{-1}\left[\sum_{i=1}^{N}(\Delta\boldsymbol{x}_{i2})(\Delta y_{i2})'\right] \\ &= \widehat{\boldsymbol{\beta}}_{FD}\end{aligned}$$

Question 3

GLS Estimation of First Differences is Equivalent to Fixed Effects

Consider the standard static linear panel data model with individual effects:

$$y_{it}=\boldsymbol{x}_{it}'\beta+\alpha_i+\varepsilon_{it} \qquad i=1,\dots,N \quad t=1,\dots,T$$

where ε_{it} is iid with variance σ^2.

Let $\boldsymbol{y}_i=(y_{i1},\dots,y_{iT})'$ and similarly for $\boldsymbol{u}_i$ and $\boldsymbol{x}_i$ (now $T\times K$). Let $\boldsymbol{\iota}_T$ be a $T\times 1$ vector of ones. The model can now be written as:

$$\boldsymbol{y}_i=\boldsymbol{x}_i\boldsymbol{\beta}+\alpha_i\boldsymbol{\iota}_T+\boldsymbol{\varepsilon}_i$$

Define the $(T-1)\times T$ matrix:

$$\boldsymbol{D}=\begin{pmatrix}1 & -1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 1 & -1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & -1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1 & -1\end{pmatrix}$$

Premultiplying the model by this matrix gives:

$$\begin{aligned} \boldsymbol{D}\boldsymbol{y}_i &= \boldsymbol{D}\boldsymbol{x}_i\boldsymbol{\beta} + \alpha_i\boldsymbol{D}\boldsymbol{\iota}_T + \boldsymbol{D}\boldsymbol{\varepsilon}_i \\ \boldsymbol{D}\boldsymbol{y}_i &= \boldsymbol{D}\boldsymbol{x}_i\boldsymbol{\beta} + \boldsymbol{D}\boldsymbol{\varepsilon}_i \end{aligned}$$

Each row of the above is now:

$$\Delta y_{it} = \Delta\boldsymbol{x}_{it}'\boldsymbol{\beta} + \Delta\varepsilon_{it}$$

The matrix $\boldsymbol{D}$ transforms the model from levels to first differences. If we estimate the transformed model with OLS, we get:

$$\begin{aligned} \widehat{\boldsymbol{\beta}}_{FD} &= \left(\sum_{i=1}^{N} (\boldsymbol{D}\boldsymbol{x}_i)' (\boldsymbol{D}\boldsymbol{x}_i)\right)^{-1} \left(\sum_{i=1}^{N} (\boldsymbol{D}\boldsymbol{x}_i)' (\boldsymbol{D}\boldsymbol{y}_i)\right) \\ &= \left(\sum_{i=1}^{N} \boldsymbol{x}_i'\boldsymbol{D}'\boldsymbol{D}\boldsymbol{x}_i\right)^{-1} \left(\sum_{i=1}^{N} \boldsymbol{x}_i'\boldsymbol{D}'\boldsymbol{D}\boldsymbol{y}_i\right) \end{aligned}$$

The error ε_{it} is iid with variance σ^2 in the model in levels. However, taking first differences actually induces serial correlation, as $\Delta\varepsilon_{it} = \varepsilon_{it} - \varepsilon_{it-1}$ is correlated with $\Delta\varepsilon_{it-1} = \varepsilon_{it-1} - \varepsilon_{it-2}$ as both contain ε_{it-1}. Now $\text{Var}(\Delta\varepsilon_{it}) = 2\sigma^2$ and $\text{Cov}(\Delta\varepsilon_{it}, \Delta\varepsilon_{it-1}) = -\sigma^2$. In matrix notation:

$$\text{Var}(\boldsymbol{D}\boldsymbol{\varepsilon}_i) = \sigma^2\boldsymbol{D}\boldsymbol{D}' \equiv \sigma^2\boldsymbol{\Omega}$$

where

$$\underset{(T-1)\times(T-1)}{\boldsymbol{\Omega}} = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 \end{pmatrix}$$

Given this, the OLS estimator $\widehat{\boldsymbol{\beta}}_{FD}$ is not efficient (although it is still consistent). To obtain an efficient estimator, we could estimate the model with GLS. What is interesting here that we know $\boldsymbol{\Omega}$ exactly as it just zeros, (negative) ones and twos, so we don't have to do FGLS.

Question: Show that the GLS estimator of this model is the same as the within estimator.

Solution

To estimate the model via GLS, we premultiply the first-differenced model by $\boldsymbol{\Omega}^{-\frac{1}{2}}$:

$$\begin{aligned} \boldsymbol{\Omega}^{-\frac{1}{2}}\boldsymbol{D}\boldsymbol{y}_i &= \boldsymbol{\Omega}^{-\frac{1}{2}}\boldsymbol{D}\boldsymbol{x}_i\boldsymbol{\beta} + \boldsymbol{\Omega}^{-\frac{1}{2}}\alpha_i\boldsymbol{D}\boldsymbol{\iota}_T + \boldsymbol{\Omega}^{-\frac{1}{2}}\boldsymbol{D}\boldsymbol{\varepsilon}_i \\ &= \boldsymbol{\Omega}^{-\frac{1}{2}}\boldsymbol{D}\boldsymbol{x}_i\boldsymbol{\beta} + \boldsymbol{\Omega}^{-\frac{1}{2}}\boldsymbol{D}\boldsymbol{\varepsilon}_i \end{aligned}$$

The GLS estimator of the first differenced model is then:

$$\begin{aligned}\widehat{\boldsymbol{\beta}}_{FD,GLS} &= \left(\sum_{i=1}^{N} \boldsymbol{x}_i' \boldsymbol{D}' \boldsymbol{\Omega}^{-1} \boldsymbol{D} \boldsymbol{x}_i\right)^{-1} \left(\sum_{i=1}^{N} \boldsymbol{x}_i' \boldsymbol{D}' \boldsymbol{\Omega}^{-1} \boldsymbol{D} \boldsymbol{y}_i\right) \\ &= \left(\sum_{i=1}^{N} \boldsymbol{x}_i' \boldsymbol{D}' \left(\boldsymbol{D}\boldsymbol{D}'\right)^{-1} \boldsymbol{D} \boldsymbol{x}_i\right)^{-1} \left(\sum_{i=1}^{N} \boldsymbol{x}_i' \boldsymbol{D}' \left(\boldsymbol{D}\boldsymbol{D}'\right)^{-1} \boldsymbol{D} \boldsymbol{y}_i\right)\end{aligned}$$

We now write the fixed effects estimator in the same notation. The mean of the $T \times 1$ vector $\boldsymbol{y}_i$ can be found by $\frac{1}{T}\boldsymbol{\iota}_T'\boldsymbol{y}_T$. To repeat the mean T times in a $T \times 1$ vector, we can write $\frac{1}{T}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\boldsymbol{y}_T$. Note that $\boldsymbol{\iota}_T\boldsymbol{\iota}_T'$ is a $T \times T$ matrix of ones. The demeaning of $\boldsymbol{y}_i$ in the fixed effects approach can therefore be done in matrix notation with:

$$\left(\boldsymbol{I}_T - \frac{1}{T}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\right)\boldsymbol{y}_i$$

Define the demeaning matrix as $\boldsymbol{Q} = \boldsymbol{D}'\left(\boldsymbol{D}\boldsymbol{D}'\right)^{-1}\boldsymbol{D} = \left(\boldsymbol{I}_T - \frac{1}{T}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\right)$. Premultiplying the model by $\boldsymbol{Q}$ demeans each variable:

$$\begin{aligned}\boldsymbol{Q}\boldsymbol{y}_i &= \boldsymbol{Q}\boldsymbol{x}_i\boldsymbol{\beta} + \boldsymbol{Q}\alpha_i\boldsymbol{\iota}_T + \boldsymbol{Q}\boldsymbol{\varepsilon}_i \\ \boldsymbol{Q}\boldsymbol{y}_i &= \boldsymbol{Q}\boldsymbol{x}_i\boldsymbol{\beta} + \left(\boldsymbol{I}_T - \frac{1}{T}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\right)\alpha_i\boldsymbol{\iota}_T + \boldsymbol{Q}\boldsymbol{\varepsilon}_i \\ \boldsymbol{Q}\boldsymbol{y}_i &= \boldsymbol{Q}\boldsymbol{x}_i\boldsymbol{\beta} + \alpha_i\boldsymbol{\iota}_T + \alpha_i\frac{1}{T}\boldsymbol{\iota}_T\underbrace{\boldsymbol{\iota}_T'\boldsymbol{\iota}_T}_{=T} + \boldsymbol{Q}\boldsymbol{\varepsilon}_i \\ \boldsymbol{Q}\boldsymbol{y}_i &= \boldsymbol{Q}\boldsymbol{x}_i\boldsymbol{\beta} + \boldsymbol{Q}\boldsymbol{\varepsilon}_i\end{aligned}$$

We can then estimate the demeaned model with OLS:

$$\widehat{\boldsymbol{\beta}}_{FE} = \left(\sum_{i=1}^{N} \boldsymbol{x}_i'\boldsymbol{Q}'\boldsymbol{Q}\boldsymbol{x}_i\right)^{-1}\left(\sum_{i=1}^{N} \boldsymbol{x}_i'\boldsymbol{Q}'\boldsymbol{Q}\boldsymbol{y}_i\right)$$

$\boldsymbol{Q}$ is symmetric:

$$\boldsymbol{Q}' = \left[\boldsymbol{D}'\left(\boldsymbol{D}\boldsymbol{D}'\right)^{-1}\boldsymbol{D}\right]' = \boldsymbol{D}'\left(\boldsymbol{D}\boldsymbol{D}'\right)^{-1}\boldsymbol{D} = \boldsymbol{Q}$$

and idempotent:

$$\boldsymbol{Q}\boldsymbol{Q} = \boldsymbol{D}'\left(\boldsymbol{D}\boldsymbol{D}'\right)^{-1}\boldsymbol{D}\boldsymbol{D}'\left(\boldsymbol{D}\boldsymbol{D}'\right)^{-1}\boldsymbol{D} = \boldsymbol{D}'\left(\boldsymbol{D}\boldsymbol{D}'\right)^{-1}\boldsymbol{D} = \boldsymbol{Q}$$

so $\boldsymbol{Q}'\boldsymbol{Q} = \boldsymbol{Q}$. Therefore:

$$\widehat{\boldsymbol{\beta}}_{FE} = \left(\sum_{i=1}^{n} \boldsymbol{x}_i'\boldsymbol{Q}\boldsymbol{x}_i\right)^{-1}\left(\sum_{i=1}^{n} \boldsymbol{x}_i'\boldsymbol{Q}\boldsymbol{y}_i\right)$$

But this is exactly the same as the GLS estimator of the first differences model.

Question 4

Staggered Difference-in-Differences

Different regions in a country are rolling out a new policy in a staggered way, with some regions rolling out the policy earlier than others. You are interested in estimating the effect of this policy on an outcome Y_{st}

in the model:

$$Y_{st} = \alpha_s + \lambda_t + \beta D_{st} + \varepsilon_{st}$$

where s indexes the region, t indexes time, α_s are region fixed effects, λ_t are time period fixed effects, and D_{st} is a treatment status indicator. Here, $D_{st} = 1$ if region s rolled out a policy in or before time t and zero otherwise. Thus, if region s is treated at time t, it remains treated in all future time periods. It holds that:

- If $D_{st} = 1$, then $D_{st'} = 1$ for all $t' \geq t$
- If $D_{st} = 0$, then $D_{st'} = 0$ for all $t' \leq t$

You have a balanced panel of S regions and T time periods.

Define $Y_{st}(1)$ as the potential outcome for region s in time t if it is treated and $Y_{st}(0)$ if it is not treated. Assume the following:

1. Observations are mutually independent across regions.
2. The potential outcomes without treatment are mean independent of the treatment sequence:

$$\mathbb{E}\left[Y_{st}(0) - Y_{st-1}(0) \,|D_{s1}, \ldots, D_{sT}\right] = \mathbb{E}\left[Y_{st}(0) - Y_{st-1}(0)\right] \text{ for all } g \text{ and } t \geq 2$$

3. Common trends in the absence of treatment: For $t \geq 2$, $\mathbb{E}\left[Y_{st}(0) - Y_{st-1}(0)\right]$ does not vary across s.

Note: It is possible to answer each part in without having solved the previous part(s). Therefore if you cannot solve a part, move on to the next one.

(i) Show that the two-way fixed effects estimator for β can be written in the form:

$$\widehat{\beta} = \frac{\sum_{s=1}^{S} \sum_{t=1}^{T} \widetilde{D}_{st} Y_{st}}{\sum_{s=1}^{S} \sum_{t=1}^{T} \widetilde{D}_{st} D_{st}}$$

where:

$$\widetilde{D}_{st} = D_{st} - \frac{1}{T}\sum_{t=1}^{T} D_{st} - \frac{1}{S}\sum_{s=1}^{S} D_{st} + \frac{1}{ST}\sum_{s=1}^{S}\sum_{t=1}^{T} D_{st}$$

Hint: First use the Frisch-Waugh-Lovell (FWL) theorem and then show that the residuals from regressing the treatment status indicator on region and time period fixed effects are equal to $\widetilde{D}_{st}$ (by taking first-order conditions of the ordinary least squares minimization problem associated with that regression).

(ii) Let $\boldsymbol{D} = (D_{11}, D_{12}, \ldots, D_{ST})$. Using:

$$\begin{aligned}\mathbb{E}\left[Y_{st}|\boldsymbol{D}\right] =& \mathbb{E}\left[Y_{st}(0)\,|\boldsymbol{D}\right] + D_{st}\mathbb{E}\left[Y_{st}(1) - Y_{st}(0)\,|\boldsymbol{D}\right] \\ =& \mathbb{E}\left[Y_{st}(0)\,|\boldsymbol{D}\right] + D_{st}\mathbb{E}\left[\Delta_{st}|\boldsymbol{D}\right]\end{aligned}$$

and assumptions 2 & 3 above, show that:

$$\begin{aligned}&\mathbb{E}\left[Y_{st}|\boldsymbol{D}\right] - \mathbb{E}\left[Y_{st'}|\boldsymbol{D}\right] - \left(\mathbb{E}\left[Y_{s't}|\boldsymbol{D}\right] - \mathbb{E}\left[Y_{s't'}|\boldsymbol{D}\right]\right) = \\ &D_{st}\mathbb{E}\left[\Delta_{st}|\boldsymbol{D}\right] - D_{st'}\mathbb{E}\left[\Delta_{st'}|\boldsymbol{D}\right] - \left(D_{s't}\mathbb{E}\left[\Delta_{s't}|\boldsymbol{D}\right] - D_{s't'}\mathbb{E}\left[\Delta_{s't'}|\boldsymbol{D}\right]\right)\end{aligned}$$

where $\mathbb{E}\left[\Delta_{st}|\boldsymbol{D}\right]$ is the average treatment effect on the treated for region s at time t.

(iii) Show that:

$$\sum_{t=1}^{T} \widetilde{D}_{st} = 0$$

By a similar approach it is possible to show that $\sum_{s=1}^{S} \widetilde{D}_{st} = 0$, which you may use later without proof.

(iv) Using parts (i)-(iii) and the assumptions above, show that in expectation the fixed effects estimator is the following weighted average of individual treatment effects:

$$\mathbb{E}\left[\widehat{\beta}\middle|\boldsymbol{D}\right] = \frac{\sum_{(s,t):D_{st}=1} \widetilde{D}_{st}\mathbb{E}\left[\Delta_{st}|\boldsymbol{D}\right]}{\sum_{(s',t'):D_{s't'}=1} \widetilde{D}_{s't'}}$$

Hint: Use the result from part (iii) that:

$$\sum_{s=1}^{S}\sum_{t=1}^{T} \widetilde{D}_{st}\mathbb{E}\left[Y_{st}|\,\boldsymbol{D}\right] = \sum_{s=1}^{S}\sum_{t=1}^{T} \widetilde{D}_{st}\left(\mathbb{E}\left[Y_{st}|\,\boldsymbol{D}\right] - \mathbb{E}\left[Y_{st'}|\,\boldsymbol{D}\right] - \mathbb{E}\left[Y_{s't}|\,\boldsymbol{D}\right] + \mathbb{E}\left[Y_{s't'}|\,\boldsymbol{D}\right]\right)$$

because the 2nd, 3rd and 4th terms equal zero when multiplied by $\widetilde{D}_{st}$ and summed over s and t.

(v) **[5 Points]** In 1998, Israel extended the right to counsel to suspects in arrest proceedings, as well as providing public defenders in arrest proceedings. They rolled out this policy change across all six regions in the country in a staggered way:

- November 1998: Tel Aviv Region and Central Region.
- January 1999: Jerusalem Region and Southern Region.
- December 2000: Northern Region.
- November 2002: Haifa Region.

A 2017 article in the *Americal Economic Journal: Economic Policy* studies the effect of this reform on crime. They use weekly data at the region level and regression 1 in Table 4 estimates the model:

$$\log\left(\text{Number of arrests}_{st}\right) = \beta\left(\text{Counsel}\right)_{st} + \alpha_s + \lambda_t + \varepsilon_{st}$$

The dependent variable is the number of arrests in region s in week t. The variable "Counsel" is an indicator if region s had implemented the reform by week t. They estimate the model using weekly data at the region level from 1996-2003 using the two-way fixed effects estimator. Their results are shown below:

TABLE 4—EFFECT OF REFORM ON THE NUMBER OF ARRESTS

Dep. variable:	log (number of arrests)		log (number of court approved arrests)	
	(1)	(2)	(3)	(4)
Right to counsel	−0.0570	−0.0486	−0.156	−0.143
	(0.0206)	(0.0206)	(0.0275)	(0.0283)
Week/region fixed effects	✓	✓	✓	✓
Region-specific time trend		✓		✓
Observations	2,496	2,496	2,496	2,496
R^2	0.785	0.79	0.622	0.631

Notes: The unit of observation is a region-week cell. Standard errors are robust and clustered by region-month.

From this we could make the interpretation that police are more hesitant to make arrests when they know they will face a public defender in court. So they make fewer arrests. Note that 8 years $\times$ 52 weeks $\times$ 6 regions gives 2,496 observations, so they have a balanced panel.

If we instead use the Callaway and Sant'Anna (2020) approach to estimate the effect of the treatment, we obtain an estimate of 0.0051 (different sign) with standard error 0.0621 for the equivalent regression in column (1). Furthermore, we can also calculate that 23.2% of the $\widetilde{D}_{st}$ terms are negative when $D_{st} = 1$.

Using the result from part (iv), discuss possible reasons why the results from using the Callaway and Sant'Anna (2020) approach would differ so much from the two-way fixed effects estimator. Come up with a plausible explanation using the distribution of treatment timing and the sample period.

Solution

(i) By the Frisch-Waugh-Lovell theorem, if we regress the treatment status indicator D_{st} on the region and time fixed effects and get the residuals u_{st}, then we can regress Y_{st} on these residuals to obtain the OLS estimator for β:

$$\widehat{\beta} = \frac{\sum_{s=1}^{S} \sum_{t=1}^{T} u_{st} Y_{st}}{\sum_{s=1}^{S} \sum_{t=1}^{T} u_{st} D_{st}}$$

We just need to show that $u_{st} = \widetilde{D}_{st}$.

Consider this first-stage regression:

$$D_{st} = \alpha_s + \lambda_t + u_{st}$$

where instead of forcing $\lambda_1 = 0$ we impose the constraint that $\sum_{t=1}^{T} \lambda_t = 0$. Ordinary least squares solves the following problem:

$$\min_{\{\alpha_1,\ldots,\alpha_S,\lambda_1,\ldots,\lambda_T\}} \sum_{s=1}^{S} \sum_{t=1}^{T} (D_{st} - \alpha_s - \lambda_t)^2$$

The first-order condition with respect to α_s is:

$$-2\sum_{t=1}^{T}(D_{st}-\alpha_s-\lambda_t)=0$$

Solving for α_s gives:

$$\alpha_s=\frac{1}{T}\sum_{t=1}^{T}(D_{st}-\lambda_t)=\frac{1}{T}\sum_{t=1}^{T}D_{st}$$

as $\sum_{t=1}^{T}\lambda_t=0$. The first-order condition with respect to λ_t is:

$$-2\sum_{s=1}^{S}(D_{st}-\alpha_s-\lambda_t)=0$$

Solving for λ_t gives

$$\lambda_t=\frac{1}{S}\sum_{s=1}^{S}D_{st}-\frac{1}{S}\sum_{s=1}^{S}\alpha_s=\frac{1}{S}\sum_{s=1}^{S}D_{st}-\frac{1}{S}\sum_{s=1}^{S}\frac{1}{T}\sum_{t=1}^{T}D_{st}$$

So the residuals from this regression can be written as:

$$\begin{aligned}u_{st}&=D_{st}-\alpha_s-\lambda_t\\u_{st}&=D_{st}-\frac{1}{T}\sum_{t=1}^{T}D_{st}-\frac{1}{S}\sum_{s=1}^{S}D_{st}+\frac{1}{ST}\sum_{s=1}^{S}\sum_{t=1}^{T}D_{st}\end{aligned}$$

(ii) Clearly we just need to show that:

$$\mathbb{E}\left[Y_{st}(0)\,|\boldsymbol{D}\right]-\mathbb{E}\left[Y_{st'}(0)\,|\boldsymbol{D}\right]-\left(\mathbb{E}\left[Y_{s't}(0)\,|\boldsymbol{D}\right]-\mathbb{E}\left[Y_{s't'}(0)\,|\boldsymbol{D}\right]\right)=0$$

Using assumption 2, this is:

$$\mathbb{E}\left[Y_{st}(0)\right]-\mathbb{E}\left[Y_{st'}(0)\right]-\left(\mathbb{E}\left[Y_{s't}(0)\right]-\mathbb{E}\left[Y_{s't'}(0)\right]\right)$$

And using assumption 3 of common trends, we know that

$$\mathbb{E}\left[Y_{st}(0)\right]-\mathbb{E}\left[Y_{st'}(0)\right]=\mathbb{E}\left[Y_{s't}(0)\right]-\mathbb{E}\left[Y_{s't'}(0)\right]$$

Thus we have shown that the expression is zero.

(iii)

$$\begin{aligned}\sum_{t=1}^{T}\widetilde{D}_{st}=&\sum_{t=1}^{T}D_{st}-\sum_{t=1}^{T}\frac{1}{T}\sum_{t=1}^{T}D_{st}-\sum_{t=1}^{T}\frac{1}{S}\sum_{s=1}^{S}D_{st}+\sum_{t=1}^{T}\frac{1}{ST}\sum_{s=1}^{S}\sum_{t=1}^{T}D_{st}\\=&-\sum_{t=1}^{T}\frac{1}{S}\sum_{s=1}^{S}D_{st}+\frac{1}{S}\sum_{s=1}^{S}\sum_{t=1}^{T}D_{st}\\=&0\end{aligned}$$

(iv) First note from part (i) that:

$$\mathbb{E}\left[\widehat{\beta}\middle|\,\boldsymbol{D}\right] = \mathbb{E}\left[\frac{\sum_{s=1}^{S}\sum_{t=1}^{T}\widetilde{D}_{st}Y_{st}}{\sum_{s=1}^{S}\sum_{t=1}^{T}\widetilde{D}_{st}D_{st}}\middle|\,\boldsymbol{D}\right] = \frac{\sum_{s=1}^{S}\sum_{t=1}^{T}\widetilde{D}_{st}\mathbb{E}\left[Y_{st}|\,\boldsymbol{D}\right]}{\sum_{s=1}^{S}\sum_{t=1}^{T}\widetilde{D}_{st}D_{st}}$$

The denominator follows from part (i):

$$\sum_{s=1}^{S}\sum_{t=1}^{T}\widetilde{D}_{st}D_{st} = \sum_{(s',t'):D_{s't'}=1}\widetilde{D}_{s't'}$$

So we only need to show in the numerator that:

$$\sum_{s=1}^{S}\sum_{t=1}^{T}\widetilde{D}_{st}\mathbb{E}\left[Y_{st}|\,\boldsymbol{D}\right] = \sum_{s=1}^{S}\sum_{t=1}^{T}\widetilde{D}_{st}D_{st}\mathbb{E}\left[\Delta_{st}|\,\boldsymbol{D}\right] =$$

Using part (iii), we can write:

$$\sum_{s=1}^{S}\sum_{t=1}^{T}\widetilde{D}_{st}\mathbb{E}\left[Y_{st}|\,\boldsymbol{D}\right] = \sum_{s=1}^{S}\sum_{t=1}^{T}\widetilde{D}_{st}\left(\mathbb{E}\left[Y_{st}|\,\boldsymbol{D}\right] - \mathbb{E}\left[Y_{st'}|\,\boldsymbol{D}\right] - \mathbb{E}\left[Y_{s't}|\,\boldsymbol{D}\right] + \mathbb{E}\left[Y_{s't'}|\,\boldsymbol{D}\right]\right)$$

That is, parts 2-4 on the RHS equal zero so we can add and subtract the terms as we please. Using part (ii)

$$\begin{aligned}&\sum_{s=1}^{S}\sum_{t=1}^{T}\widetilde{D}_{st}\mathbb{E}\left[Y_{st}|\,\boldsymbol{D}\right]\\ =&\sum_{s=1}^{S}\sum_{t=1}^{T}\widetilde{D}_{st}\left(D_{st}\mathbb{E}\left[\Delta_{st}|\boldsymbol{D}\right] - D_{st'}\mathbb{E}\left[\Delta_{st'}|\boldsymbol{D}\right] - \left(D_{s't}\mathbb{E}\left[\Delta_{s't}|\boldsymbol{D}\right] - D_{s't'}\mathbb{E}\left[\Delta_{s't'}|\boldsymbol{D}\right]\right)\right)\end{aligned}$$

Removing the parts that we know equal zero (for the same reason as the 2nd-last step):

$$\begin{aligned}&\sum_{s=1}^{S}\sum_{t=1}^{T}\widetilde{D}_{st}\mathbb{E}\left[Y_{st}|\,\boldsymbol{D}\right]\\ =&\sum_{s=1}^{S}\sum_{t=1}^{T}\widetilde{D}_{st}D_{st}\mathbb{E}\left[\Delta_{st}|\boldsymbol{D}\right]\\ =&\sum_{(s,t):D_{st}=1}\widetilde{D}_{st}\mathbb{E}\left[\Delta_{st}|\boldsymbol{D}\right]\end{aligned}$$

And we are done.

(v) We saw in part (iv) that the two-way fixed effects estimator of β is a weighted average of the individual treatment effects. These weights are proportional to $\widetilde{D}_{st}$, which are more likely to be negative when:

- The region is an early adopter. Here the earliest adopter have almost 3 years of pre-treatment data, so this problem is less likely of a concern.
- The time period is toward the end of the sample period. This is more of a concern here because all regions are eventually treated, and 5 of the 6 are treated around the middle of the sample period.

So negative weights are likely towards the end of the sample.

If the instantaneous effect of the policy is close to zero, but eventually (after several years), the policy starts to have a positive effect on the number of arrests (perhaps because the policy leads to more crime), then negative weights on early adopters in the later years (which are the most likely to have negative weights) could lead the two-way fixed effects estimator to be negative.

Question 5

Application of Changes-in-Changes

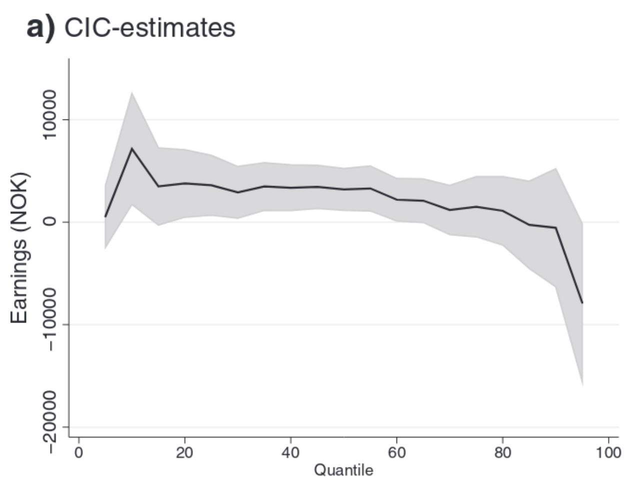
Havnes and Mogstad (2015)[1] use a reform which introduced universal subsidized childcare for 3- to 6-year olds in Norway to study the effect of childcare on future earnings as an adult.

The reform occurred during 1976-1979. The *post-reform cohort* (those affected by the reform) is those born during 1973-1976 and the *pre-reform cohort* (those unaffected by the reform) is those born during 1967-1969. They order municipalities by how much they grew (in percentage points) in childcare coverage between 1976 and 1979. They split municipalities at the median and the *treated municipalities* are the municipalities that experienced an above-the-median growth in childcare coverage, and the *control municipalities* are the municipalities that experienced a below-the-median growth in childcare coverage.

The empirical strategy is then to compare later adult outcomes (such as earnings) of the children from treated and control municipalities before and after the reform in a differences-in-differences setup.

(i) What concerns could you have about this empirical strategy?

(ii) The changes-in-changes estimate at each quantile in the earnings distribution is shown below. The grey band is a 90% confidence interval from 500 bootstrap replications. The earnings variable is defined as the individual's average earnings over the period 2006-2009 measured in Norwegian Kroner. Explain and interpret what the graph tells us.



(iii) Havenes and Mogstad (2015) also find that the policy had a positive effect on the child's future earnings when they had low-income parents and had a negative effect when they had middle- to high-income parents. What are the policy implications of this?

[1]Havnes, Tarjei, and Magne Mogstad "Is universal child care leveling the playing field?." *Journal of Public Economics* 127 (2015): 100-114.

Example Solution

(i) One problem would be if there is a trend in an unobserved factor that affected both childcare coverage and later earnings. An example of this would be a trend in parents' educational attainment. This would increase the demand for childcare (as it's more likely both will be working) but could also have impacts on children's later earnings (as parents' education may affect the children's educational outcomes).

There are also concerns about selection into the treatment group: the municipalities that saw larger increases in childcare coverage may have been the ones that would benefit the most from increasing coverage.

(ii) The policy had a positive effect for children in the low and middle parts of the income distribution. For the upper part of the distribution, the effect is negative, although not statistically significant. Also, since the incomes at the lower end of the distribution increased and did not increase at the upper end, it had a small effect of reducing income inequality.

(iii) Given this, providing childcare to upper-class families may not be worth the cost. Therefore a means-tested subsidy for childcare may be more cost-effective.