In Step 8 of the second welfare theorem proof, we argued that if $\boldsymbol{x}_i' \succ_i \boldsymbol{x}_i^\star$, that $\alpha \boldsymbol{x}_i' \succ_i \boldsymbol{x}_i^\star$ for an $\alpha \in (0,1)$ sufficiently close to 1.

Proof: We define the sets:

$$\mathcal{A}^+ = \left\{\gamma \in [0,1] : \gamma \boldsymbol{x}_i' \succeq_i \boldsymbol{x}_i^\star\right\}$$
$$\mathcal{A}^- = \left\{\gamma \in [0,1] : \boldsymbol{x}_i^\star \succeq_i \gamma \boldsymbol{x}_i'\right\}$$

We break the proof up into 6 steps.

1. **Step 1:** The sets $\mathcal{A}^+$ and $\mathcal{A}^-$ are nonempty.
   - Because $\boldsymbol{x}_i' \succ_i \boldsymbol{x}_i^\star$, we have $\gamma \boldsymbol{x}_i' \succ_i \boldsymbol{x}_i^\star$ for $\gamma = 1$. Hence $\mathcal{A}^+$ is nonempty.
   - Because $\boldsymbol{x}_i^\star \gg \boldsymbol{0}$ and because $\boldsymbol{x}_i^\star$ is Pareto optimal, we have that $\boldsymbol{x}_i^\star \succeq_i \boldsymbol{0}$. This is because $\boldsymbol{0}$ was always feasible to give to i and not make anyone worse off so it cannot make them strictly better off compared to $\boldsymbol{x}_i^\star$. Therefore $\boldsymbol{x}_i^\star \succeq_i \gamma \boldsymbol{x}_i'$ for $\gamma = 0$. Hence $\mathcal{A}^-$ is nonempty.
2. **Step 2:** The sets $\mathcal{A}^+$ and $\mathcal{A}^-$ are closed.
   - We will show this for $\mathcal{A}^+$ and a similar argument holds for $\mathcal{A}^-$. Consider a sequence $\gamma_n \longrightarrow \gamma$ with each $\gamma_n \in \mathcal{A}^+$, implying $\gamma_n \in [0,1]$ and $\gamma_n \boldsymbol{x}_i' \succeq_i \boldsymbol{x}_i^\star$ for all n. By the continuity of preferences, the preference relation is preserved under limits: if $\gamma_n \boldsymbol{x}_i' \succeq_i \boldsymbol{x}_i^\star$ for all n, then $\gamma \boldsymbol{x}_i' \succeq_i \boldsymbol{x}_i^\star$. Because the unit interval is closed, we also know that $\gamma \in [0,1]$. Hence $\mathcal{A}^+$ is closed.
3. **Step 3:** $\mathcal{A}^+ \cup \mathcal{A}^- = [0,1]$.
   - This follows from the completeness of preferences $\succeq_i$. For any γ, either $\gamma \boldsymbol{x}_i' \succeq_i \boldsymbol{x}_i^\star$ or $\boldsymbol{x}_i^\star \succeq_i \gamma \boldsymbol{x}_i'$ (or both).
4. **Step 4:** $\mathcal{A}^+ \cap \mathcal{A}^- \neq \varnothing$.
   - From step 3, $\mathcal{A}^+ \cup \mathcal{A}^- = [0,1]$, a *connected* set.[1]
   - Suppose toward a contradiction that $\mathcal{A}^+ \cap \mathcal{A}^- = \varnothing$. Because $\mathcal{A}^+$ and $\mathcal{A}^-$ are both closed and disjoint, they are separated. From step 3, this means we can write $[0,1]$ as the union of two non-empty separated sets. But this contradicts that $[0,1]$ is connected.
5. **Step 5:** $\exists \beta \in [0,1)$ such that $\beta \boldsymbol{x}_i' \sim_i \boldsymbol{x}_i^\star$.
   - That such a $\beta \in [0,1]$ exists comes from the non-empty intersection of $\mathcal{A}^+ \cap \mathcal{A}^-$. We just need to rule out the possibility $\beta = 1$. But $\boldsymbol{x}_i' \succ_i \boldsymbol{x}_i^\star$ so we cannot have $\beta = 1$.
6. **Step 6:** $\exists \alpha$ satisfying $\alpha \boldsymbol{x}_i' \succ_i \boldsymbol{x}_i^\star$.
   - We know that $\boldsymbol{x}_i' \succ_i \boldsymbol{x}_i^\star$ and $\beta \boldsymbol{x}_i' \sim_i \boldsymbol{x}_i^\star$.
   - By convexity, we have $\lambda \boldsymbol{x}_i' + (1-\lambda)\beta \boldsymbol{x}_i' \succ_i \boldsymbol{x}_i^\star$ for all $\lambda \in (0,1)$.
   - Because $\lambda \in (0,1)$ and $\beta \in [0,1)$, we have $0 < \lambda + (1-\lambda)\beta < 1$ and $[\lambda + (1-\lambda)\beta]\, \boldsymbol{x}_i' \succ_i \boldsymbol{x}_i^\star$.
   - Thus $\alpha = \lambda + (1-\lambda)\beta$ satisfies $\alpha \boldsymbol{x}_i' \succ_i \boldsymbol{x}_i^\star$

[1] A connected set is a set which cannot be written as the union of two non-empty separated sets (sets where each set is disjoint from the other's closure). That is, the set X is connected if it cannot be written as the union of 2 non-empty sets A and B, where both $A \cap \overline{B} = \varnothing$ and $\overline{A} \cap B = \varnothing$.