## Equilibrium Existence and Uniqueness

We will show with the aid of diagrams that an equilibrium exists in the economy, and that it is unique. We first construct the aggregate demand and supply curves, and show that these intersect exactly once, provided $\max_i \{\phi_i'(0)\} > \min_j \{c_j'(0)\}$.

To ease notation below, we will define $\bar{c} = \max_i \{\phi_i'(0)\}$ and $\underline{c} = \min_j \{c_j'(0)\}$.

### Aggregate Demand

Consumer $i$'s optimality condition was $\phi_i'(x_i) \leq p$, with equality if $x_i > 0$. If the price is high enough such that $p > \phi_i'(0)$, then the consumer finds it optimal not to consume anything and sets $x_i = 0$. Otherwise, the optimal consumption of $x_i$ involves consuming until marginal utility equals price, i.e. $\phi_i'(x_i) = p$. Because $\phi_i' > 0$ and $\phi_i'' < 0$, the marginal utility function $\phi_i'$ is positive and strictly decreasing. Therefore it can be inverted to find the unique optimal $x_i$ given $p$. Therefore for all positive $p$, there is a unique $x_i$ satisfying the optimality condition. We call $x_i(p)$ the function that maps $p$ into an optimal $x_i$. This is consumer $i$'s demand function.

The demand function $x_i(p)$ is continuous, because $\phi_i'(x_i)$ is continuous and strictly decreasing. Furthermore, because $\phi_i'(x_i)$ is strictly decreasing, the demand function is decreasing for all $p$ and strictly decreasing for all $p \leq \phi_i'(0)$.

We show this graphically in the left panel of Figure 1. For any $p$ on the vertical axis, we can use $\phi_i'(x_i)$ to uniquely determine the optimal $x_i$ on the horizontal axis. If $p > \phi_i'(0)$, then the optimal quantity is zero. Otherwise we use the curve to find $x_i$.

Aggregate demand is then $x(p) = \sum_{i=1}^{I} x_i(p)$. If $p > \bar{c}$, then all consumers choose $x_i = 0$ and aggregate demand is zero. Because each individual demand function $x_i(p)$ is continuous, the aggregate demand function is continuous. Furthermore, because all the demand functions are decreasing, the aggregate demand function is decreasing. It is strictly decreasing for all $p \leq \bar{c}$.

The right panel of Figure 1 shows this for $I = 2$. When $p > \phi_2'(0)$, neither consumer buys any good $\ell$ and aggregate demand is zero. When the price is larger than $\phi_1'(0)$ but lower than $\phi_2'(0)$, only consumer 2 buys a positive quantity and $x(p) = x_2(p)$. For all $p < \phi_1'(0)$, both consumers buy a positive quantity. The aggregate demand function is then the sum of the two individual demand functions.

### Aggregate Supply

We can proceed in a similar way to characterize the aggregate supply function.

Firm $j$'s optimality condition is $c_j'(q_j) \geq p$ with equality if $q_j > 0$. If the price is low enough such that $p \leq c_j'(0)$, then the marginal cost of the first unit is greater than the price, and the

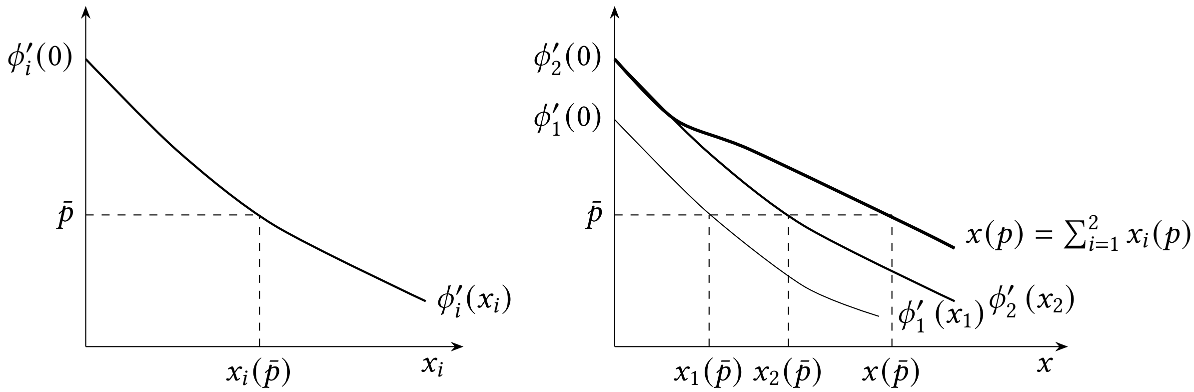


FIGURE 1: Consumer $i$'s demand function (left) and the aggregate demand function (right).

firm optimaly chooses to produce zero units (recall that the marginal cost function is strictly increasing). If $p > c'(0)$, then the firm will produce a positive amount. They will produce until marginal cost equals the price, which is at the quantity that solves $p = c'(q_j)$. Because $c_j' > 0$ and $c_j'' > 0$, $c_j'$ is positive and strictly increasing, we can invert $p = c'(q_j)$ to find the unique $q_j$ satisfying the first-order condition. This is $q_j(p)$, $j$'s supply function.

The supply function $q_j(p)$ is continuous and nondecreasing at all $p > 0$ and is strictly increasing at any $p > c_j'(0)$.

We show this graphically in the left panel of Figure 2. For any price on the vertical axis, we can use the first-order condition to uniquely determine the optimal quantity on the horizontal axis. If $p \leq c_j'(0)$, then the optimal quantity is zero. Otherwise we use the $c'(q_j)$ to find $q_j$.

Aggregate supply is then $q(p) = \sum_{j=1}^{J} q_j(p)$. $q(p) = 0$ for all $p < \underline{c}$. $q(p)$ is continuous and nondecreasing at all $p > 0$ and is strictly increasing at any $p > \underline{c}$.

We show the aggregate supply curve graphically in the right panel of Figure 2. When the price is less than or equal to $c_2'(0)$, neither firm produces and the aggregate supply is zero. When the price is between $c_2'(0)$ and $c_1'(0)$, only firm 2 produces and the aggregate supply curve corresponds to firm 2's individual supply curve. When the price exceeds $c_1'(0)$, then both firms produce and the aggregate supply curve is the sum of each individual supply curve.

**Equilibrium**

Equilibrium occurs when we have a $p^\star$ satisfying $x(p^\star) - q(p^\star) = 0$, i.e. aggregate supply equals aggregate demand.

Let's recap what we found in the previous 2 sections:

- Aggregate demand $x(p)$ is continuous and nonincreasing at all $p > 0$ and is strictly decreasing at all $p \leq \bar{c}$.

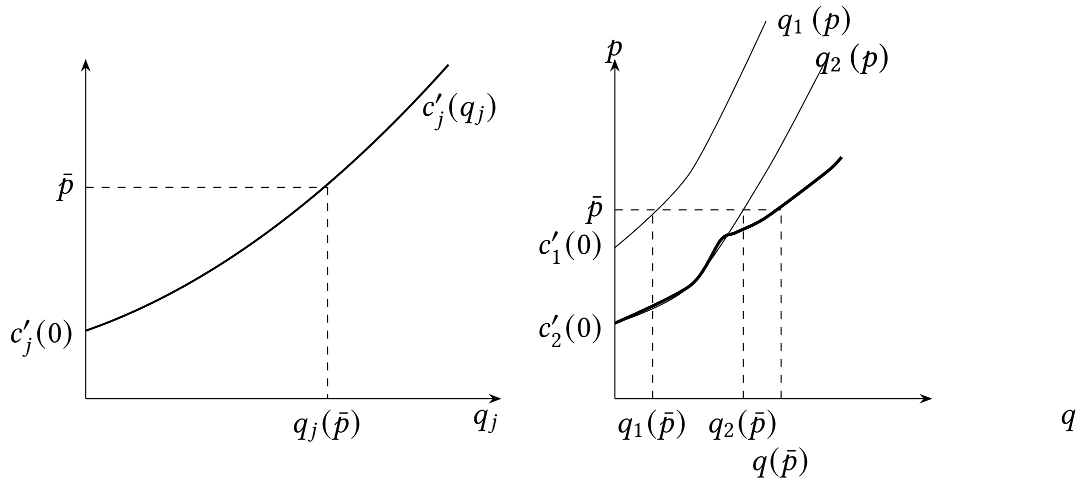


FIGURE 2: Firm $j$'s supply function (left) and the aggregate supply function (right).

- Aggregate supply $q(p)$ is continuous and nondecreasing at all $p > 0$ and is strictly increasing at all $p > \underline{c}$.

Together, this means that $x(p) - q(p)$ is continuous and nonincreasing at all $p > 0$ and is strictly decreasing at all $p \in [\underline{c}, \overline{c}]$.

We want to show that we are guaranteed the existence of an equilibrium price.

We first consider the case where $\overline{c} \leq \underline{c}$. This is the less interesting case. Here, the cost of producing the product exceeds how much consumers value consuming it, and nothing is produced or consumed in equilibrium. Any price in the interval $[\overline{c}, \underline{c}]$ is an equilibrium price vector. At any price in this interval, no consumer wishes to purchase any of the product, nor does any firm want to produce any units. Therefore aggregate supply equals aggregate demand (at zero), and we have an equilibrium.

We next consider the case where $\overline{c} > \underline{c}$. Here we make use of the intermediate value theorem. This theorem states that if a function $f$ is continuous in the interval $[a, b]$ and there is a number $c$ between the values $f(a)$ and $f(b)$, then there exists some $x \in (a, b)$ where $f(x) = c$.

Here, our $f$ will be $x(p) - q(p)$, $a$ will be $\underline{c}$, $b$ will be $\overline{c}$, and $c$ will be zero. We will use this to show that there exists some $p \in (\underline{c}, \overline{c})$ where $x(p) - q(p) = 0$.

At $p = \overline{c}$, aggregate demand is zero but aggregate supply is positive, so $x(p) - q(p) < 0$. At $p = \underline{c}$, aggregate demand is positive but aggregate supply is zero , so $x(p) - q(p) > 0$. Because $x(p) - q(p)$ is continuous and we have shown $x(\overline{c}) - q(\overline{c}) < 0 < x(\underline{c}) - q(\underline{c})$, the *existence* of a $p^\star$ satisfying $x(p^\star) - q(p^\star) = 0$ is guaranteed.

Moreover, because $x(p) - q(p)$ is a continuous and strictly decreasing over $[\underline{c}, \overline{c}]$, we are guaranteed that the equilibrium price $p^\star$ is unique.