

## A Note on Demand in the Quasilinear Model

The consumer's problem is:

$$\begin{aligned} & \max_{m_i \in \mathbb{R}, x_i \in \mathbb{R}_+} m_i + \phi_i(x_i) \\ & \text{subject to } m_i + px_i \leq \omega_{mi} + \sum_{j=1}^J \theta_{ij} (pq_j - c_j(q_j)) \end{aligned}$$

In the slides, we said the first-order conditions are:

$$\begin{cases} \phi'_i(x_i) - p \leq 0 & \text{if } x_i = 0 \\ \phi'_i(x_i) - p = 0 & \text{if } x_i > 0 \end{cases}$$

In this note, we will explain how this is derived in more detail.

Because utility is increasing in both goods, the budget constraint will hold with equality. Therefore we can transform the problem to:

$$\max_{x_i \in \mathbb{R}_+} \phi_i(x_i) - px_i + \omega_{mi} + \sum_{j=1}^J \theta_{ij} (pq_j - c_j(q_j))$$

The only constraint that remains is  $x_i \in \mathbb{R}_+$ , or alternatively stated,  $x_i \geq 0$ . The terms  $\omega_{mi} + \sum_{j=1}^J \theta_{ij} (pq_j - c_j(q_j))$  do not depend on  $x_i$ , so we can ignore them. So the problem can be restated as:

$$\max_{x_i} \phi_i(x_i) - px_i \text{ subject to } x_i \geq 0$$

In general, a univariate optimization problem with single inequality constraint is (using the notation in the Mathematical Appendix of MWG):

$$\max_x f(x) \text{ subject to } h(x) \leq c$$

The Kuhn-Tucker conditions are:  $f'(x) = \lambda h'(x)$  and  $\lambda (h(x) - c) = 0$ . This is exactly what we have here when  $h(x) = -x$  and  $c = 0$ .

So the Kuhn-Tucker conditions are:

- $\phi'_i(x_i) - p = -\lambda$
- $\lambda x_i = 0$

So:

- If  $x_i > 0$ , then the second condition requires  $\lambda = 0$ . So when  $x_i$ , the first-order condition is  $\phi'_i(x_i) - p = 0$ .

- If  $x_i = 0$ , then  $\lambda$  need not be zero. We know from Kuhn-Tucker that  $\lambda \geq 0$ , so this means  $\phi_i(x_i) - p \leq 0$ .

Putting these together:

$$\begin{cases} \phi_i'(x_i) - p \leq 0 & \text{if } x_i = 0 \\ \phi_i'(x_i) - p = 0 & \text{if } x_i > 0 \end{cases}$$

### Constraint on $m_i$

In the consumer's problem, we choose  $m_i$  from  $\mathbb{R}$  and not  $\mathbb{R}_+$ . This is done to avoid more corner solutions for  $x_i$ . In fact, this constraint could create wealth effects in the quasilinear model. Let's do the problem where we constrain the consumer to  $m_i \geq 0$ :

$$\max_{m_i \geq 0, x_i \geq 0} m_i + \phi_i(x_i) \quad \text{subject to } m_i + px_i \leq \omega_{mi} + \sum_{j=1}^J \theta_{ij} (pq_j - c_j(q_j))$$

The budget constraint will hold with equality again. Let's write the consumer's wealth as  $w_i = \omega_{mi} + \sum_{j=1}^J \theta_{ij} (pq_j - c_j(q_j))$ , so solving the budget for  $m_i$  gives  $m_i = w_i - px_i$ . The reduced problem is now:

$$\max_{x_i} \phi_i(x_i) - px_i + w_i$$

subject to:

$$\begin{aligned} x_i &\geq 0 \\ x_i &\leq \frac{w_i}{p} \end{aligned}$$

The difference here is that now the consumer needs to be able to afford their chosen quantity of  $x_i$ . Before we just ignored this part because we said they could consume negative  $m_i$  (this would be like borrowing money to finance their consumption of  $x_i$ ).

Let's find the optimal demand with a chosen utility function. Suppose  $\phi_i(x_i) = \sqrt{x_i}$ . With this utility function, there is no price high enough that would lead the consumer to choose  $x_i = 0$ , but they may hit the  $\frac{w_i}{p}$  constraint depending on wealth and prices.

At an interior solution where  $x_i \in \left(0, \frac{w_i}{p}\right)$ , the first-order condition implies:

$$\frac{1}{2\sqrt{x_i}} = p \quad \implies \quad x_i = \frac{1}{4p^2}$$

This bundle is only affordable if  $\frac{1}{4p^2} \leq \frac{w_i}{p}$ , or  $w_i \geq \frac{1}{4p}$ . So we can write the demand of consumer  $i$

for good  $x_i$  as:

$$x_i(p, w_i) = \begin{cases} \frac{1}{4p^2} & \text{if } w_i \geq \frac{1}{4p} \\ \frac{w_i}{p} & \text{if } w_i < \frac{1}{4p} \end{cases}$$

If the consumer has sufficient wealth ( $w_i \geq \frac{1}{4p}$ ), they consume their optimal amount of  $x_i$ , the same as if we had no  $m_i \geq 0$  constraint. They then spend the rest of their money on  $m_i$ . If they don't have sufficient wealth, they spend all their wealth on  $x_i$  and none on  $m_i$ . In this case, even though we have a quasilinear utility function, there are wealth effects. The demand for  $x_i$  depends on  $m_i$ .

If the market under consideration involves goods that make up a small portion of our budgets (like the purchase of pencils or erasers), then this upper bound constraint is very unlikely to be reached. To simplify the analysis we therefore assume  $m_i$  can be negative so we don't need to carry around these extra cases that complicate the analysis.

That being said, if an exam question states that  $X_i = \mathbb{R}_+^L$ , then you must work within those constraints and consider these boundary cases.