To find an equilibrium we need to find an allocation and price vector that satisfy (a) utility maximization, (b) profit maximization, and (c) market clearing in *both* goods. To help us find an equilibrium, we will first prove the following Lemma which will require us to only need to check for market clearing for good ℓ. Essentially the Lemma will show that provided prices are positive and budget constraints hold with equality, then market clearing in one good implies market clearing in the other good. This means we will only need to check for market clearing for one good.

Because we will use this Lemma in later chapters, we will do a general proof with L goods, using the general notation introduced in **??**.

Lemma 1. *If the allocation* $\left(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_I, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_J\right)$ *and price vector* $\boldsymbol{p} \gg \mathbf{0}$ *satisfy the market clearing condition for all goods* $\ell \neq k$*, and if every consumer's budget constraint is satisfied with equality, so that* $\boldsymbol{p} \cdot \boldsymbol{x}_i = \boldsymbol{p} \cdot \boldsymbol{\omega}_i + \sum_{j=1}^{J} \theta_{ij} \boldsymbol{p} \cdot \boldsymbol{y}_j$ *for all* i*, then the market for good* k *also clears.*

Proof. Consumer i's budget constraint is (imposing equality as in the Lemma):

$$\sum_{\ell=1}^{L} p_\ell x_{\ell i} - \sum_{\ell=1}^{L} p_\ell \omega_{\ell i} - \sum_{j=1}^{J} \theta_{ij} \sum_{\ell=1}^{L} p_\ell y_{\ell j} = 0$$

Summing over i:

$$\sum_{i=1}^{I} \sum_{\ell=1}^{L} p_\ell x_{\ell i} - \sum_{i=1}^{I} \sum_{\ell=1}^{L} p_\ell \omega_{\ell i} - \sum_{i=1}^{I} \sum_{j=1}^{J} \theta_{ij} \sum_{\ell=1}^{L} p_\ell y_{\ell j} = 0$$

Rearranging summation operators and simplifying terms:

$$\begin{aligned}
\sum_{\ell=1}^{L} \sum_{i=1}^{I} p_\ell x_{\ell i} - \sum_{\ell=1}^{L} \sum_{i=1}^{I} p_\ell \omega_{\ell i} - \sum_{\ell=1}^{L} \sum_{j=1}^{J} \sum_{i=1}^{I} \theta_{ij} p_\ell y_{\ell j} &= 0 \\
\sum_{\ell=1}^{L} p_\ell \sum_{i=1}^{I} x_{\ell i} - \sum_{\ell=1}^{L} p_\ell \underbrace{\sum_{i=1}^{I} \omega_{\ell i}}_{=\bar{\omega}_\ell} - \sum_{\ell=1}^{L} p_\ell \sum_{j=1}^{J} y_{\ell j} \underbrace{\sum_{i=1}^{I} \theta_{ij}}_{=1} &= 0 \\
\sum_{\ell=1}^{L} p_\ell \left(\sum_{i=1}^{I} x_{\ell i} - \bar{\omega}_\ell - \sum_{j=1}^{J} y_{\ell j} \right) &= 0
\end{aligned}$$

Splitting the sum over ℓ into k and all other goods, and moving the kth term to the right-hand side:

$$\sum_{\ell \neq k}^{L} p_\ell \left(\sum_{i=1}^{I} x_{\ell i} - \bar{\omega}_\ell - \sum_{j=1}^{J} y_{\ell j} \right) = -p_k \left(\sum_{i=1}^{I} x_{k i} - \bar{\omega}_k - \sum_{j=1}^{J} y_{k j} \right)$$

According to the statement in the Lemma, we have market clearing in all goods $\ell \neq k$. This means that $\sum_{i=1}^{I} x_{\ell i} - \bar{\omega}_\ell - \sum_{j=1}^{J} y_{\ell j} = 0$ for all goods $\ell \neq k$, and the left-hand side must be equal to zero. We also have $p_k > 0$. Therefore, for the right-hand side to equal zero, we must have $\sum_{i=1}^{I} x_{ki} - \bar{\omega}_k - \sum_{j=1}^{J} y_{kj} = 0$. But this implies market clearing for good k as well. □

Using the Lemma, we can state a formal definition for an equilibrium in this economy. The allocation $\left(x_1^\star, \ldots, x_I^\star, q_1^\star, \ldots, q_I^\star\right)$ and price $p^\star$ constitute a competitive equilibrium if and only if we have:

1. $p^\star \leq c_j'\left(q_j^\star\right)$, with equality if $q_j^\star > 0$, for all $j = 1, \ldots, J$.
2. $\phi_i'\left(x_i^\star\right) \leq p^\star$, with equality if $x_i^\star > 0$, for all $i = 1, \ldots, I$.
3. $\sum_{i=1}^{I} x_i^\star = \sum_{j=1}^{J} q_j^\star$.

Notice that point 3 only refers to market clearing in good ℓ. Market clearing in the composite commodity will follow directly based on our Lemma. Furthermore, if $\max_i \left\{\phi_i'(0)\right\} > \min_j \left\{c_j'(0)\right\}$ we will have $\sum_{i=1}^{I} x_i^\star > 0$ in equilibrium. We will assume that this is the case in what follows.