

## Note on the continuity of the fixed-point function in the existence proof

In our proof of the existence of equilibrium we said that if  $z_\ell(\mathbf{p})$ , then the function:  $f : \Delta \rightarrow \Delta$ :

$$f(\mathbf{p}) = \{f_\ell(\mathbf{p})\}_{\ell=1}^L = \left\{ \frac{p_\ell + \max\{0, z_\ell(\mathbf{p})\}}{1 + \sum_{k=1}^K \max\{0, z_k(\mathbf{p})\}} \right\}_{\ell=1}^L$$

where:

$$\Delta = \left\{ \mathbf{p} \in \mathbb{R}_+^L : \sum_{\ell=1}^L p_\ell = 1 \right\}$$

is also continuous. We gave some informal argumentation as to why this should be, namely, that both the numerator and denominator of the fraction is continuous, the denominator is bounded away from zero (because of the +1), so the whole fraction is continuous. In this note, I give a formal proof of the continuity of  $f(\mathbf{p})$ .

**Proposition 1.** *If each  $z_\ell(\mathbf{p})$  is continuous for all  $\mathbf{p} \in \Delta$ , then  $f(\mathbf{p})$  is continuous at any  $\mathbf{p} \in \Delta$ . That is, for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that  $\|\mathbf{p}' - \mathbf{p}\| < \delta$  implies  $\|f(\mathbf{p}') - f(\mathbf{p})\| < \varepsilon$ .*

*Proof.* We break the proof up into a sequence of steps. We write the fraction as  $f_\ell(\mathbf{p}) = \frac{g_\ell(\mathbf{p})}{h(\mathbf{p})}$  and show separately that  $g_\ell(\mathbf{p})$  and  $h(\mathbf{p})$  are continuous and  $h(\mathbf{p})$  is bounded away from zero. With these results we then show that the entire fraction is continuous.

STEP 1:  $g_\ell(\mathbf{p}) = p_\ell + \max\{0, z_\ell(\mathbf{p})\}$  is continuous.

We first write:

$$\begin{aligned} & \|g_\ell(\mathbf{p}') - g_\ell(\mathbf{p})\| \\ &= \|(p'_\ell + \max\{0, z_\ell(\mathbf{p}')\}) - (p_\ell + \max\{0, z_\ell(\mathbf{p})\})\| \\ &= \|p'_\ell - p_\ell + \max\{0, z_\ell(\mathbf{p}')\} - \max\{0, z_\ell(\mathbf{p})\}\| \end{aligned}$$

For  $\max\{0, z_\ell(\mathbf{p}')\} - \max\{0, z_\ell(\mathbf{p})\}$  there are 4 possible cases:

- $z_\ell(\mathbf{p}') \leq 0$  and  $z_\ell(\mathbf{p}) \leq 0 \implies \max\{0, z_\ell(\mathbf{p}')\} - \max\{0, z_\ell(\mathbf{p})\} = 0$ .
- $z_\ell(\mathbf{p}') > 0$  and  $z_\ell(\mathbf{p}) \leq 0 \implies \max\{0, z_\ell(\mathbf{p}')\} - \max\{0, z_\ell(\mathbf{p})\} = z_\ell(\mathbf{p}')$ .
- $z_\ell(\mathbf{p}') \leq 0$  and  $z_\ell(\mathbf{p}) > 0 \implies \max\{0, z_\ell(\mathbf{p}')\} - \max\{0, z_\ell(\mathbf{p})\} = -z_\ell(\mathbf{p})$ .
- $z_\ell(\mathbf{p}') > 0$  and  $z_\ell(\mathbf{p}) > 0 \implies \max\{0, z_\ell(\mathbf{p}')\} - \max\{0, z_\ell(\mathbf{p})\} = z_\ell(\mathbf{p}') - z_\ell(\mathbf{p})$ .

For small changes  $\mathbf{p}$  to  $\mathbf{p}'$ , however,  $z_\ell(\mathbf{p}')$  and  $z_\ell(\mathbf{p})$  will have the same sign (by the

continuity of  $z_\ell(\mathbf{p})$ ). Therefore for small  $\|\mathbf{p}' - \mathbf{p}\|$  we can write:

$$\begin{aligned} & \|g_\ell(\mathbf{p}') - g_\ell(\mathbf{p})\| \\ &= \|\mathbf{p}'_\ell + \max\{0, z_\ell(\mathbf{p}')\} - (\mathbf{p}_\ell + \max\{0, z_\ell(\mathbf{p})\})\| \\ &= \|\mathbf{p}'_\ell - \mathbf{p}_\ell + \max\{0, z_\ell(\mathbf{p}') - z_\ell(\mathbf{p})\}\| \end{aligned}$$

By the continuity of  $z_\ell(\mathbf{p})$ , for all  $\varepsilon > 0$  there exists a  $\tilde{\delta}_\ell > 0$  such that  $\|\mathbf{p}' - \mathbf{p}\| < \tilde{\delta}_\ell$  implies that  $\|z_\ell(\mathbf{p}') - z_\ell(\mathbf{p})\| < \frac{\varepsilon}{2}$ . This further implies that  $\|\max\{0, z_\ell(\mathbf{p}') - z_\ell(\mathbf{p})\}\| < \frac{\varepsilon}{2}$ . Moreover,  $\|\mathbf{p}' - \mathbf{p}\| < \tilde{\delta}_\ell$  implies that  $\|\mathbf{p}'_\ell - \mathbf{p}_\ell\| < \tilde{\delta}_\ell$ .<sup>1</sup> By the triangle inequality:

$$\|\mathbf{p}'_\ell - \mathbf{p}_\ell + \max\{0, z_\ell(\mathbf{p}') - z_\ell(\mathbf{p})\}\| \leq \underbrace{\|\mathbf{p}'_\ell - \mathbf{p}_\ell\|}_{< \tilde{\delta}_\ell} + \underbrace{\|\max\{0, z_\ell(\mathbf{p}') - z_\ell(\mathbf{p})\}\|}_{< \varepsilon/2}$$

Take  $\delta_\ell = \min\{\tilde{\delta}_\ell, \varepsilon/2\}$ . Then:  $\|\mathbf{p}' - \mathbf{p}\| < \delta_\ell$  implies that:

$$\|g_\ell(\mathbf{p}') - g_\ell(\mathbf{p})\| \leq \underbrace{\|\mathbf{p}'_\ell - \mathbf{p}_\ell\|}_{< \delta_\ell \leq \varepsilon/2} + \underbrace{\|\max\{0, z_\ell(\mathbf{p}') - z_\ell(\mathbf{p})\}\|}_{< \varepsilon/2} < \varepsilon$$

Therefore  $g_\ell(\mathbf{p})$  is continuous. This holds for all  $\ell = 1, \dots, L$ .

STEP 2:  $h(\mathbf{p}) = 1 + \sum_{k=1}^L \max\{0, z_k(\mathbf{p})\}$  is continuous.

Here we use that  $h(\mathbf{p}) = \sum_{\ell=1}^L g_\ell(\mathbf{p})$ , the sum of continuous functions. By the continuity of each  $g_\ell(\mathbf{p})$ , for all  $\varepsilon > 0$ , there exists a  $\delta_\ell > 0$  such that  $\|\mathbf{p}' - \mathbf{p}\| < \delta_\ell$  implies that  $\|g_\ell(\mathbf{p}') - g_\ell(\mathbf{p})\| < \varepsilon/L$ . Take  $\delta = \min\{\delta_1, \dots, \delta_L\}$ . By iterative use of the triangle inequality, for  $\|\mathbf{p}' - \mathbf{p}\| < \delta$ :

$$\|h(\mathbf{p}') - h(\mathbf{p})\| = \left\| \sum_{\ell=1}^L (g_\ell(\mathbf{p}') - g_\ell(\mathbf{p})) \right\| \leq \sum_{\ell=1}^L \underbrace{\|g_\ell(\mathbf{p}') - g_\ell(\mathbf{p})\|}_{< \varepsilon/L} < \varepsilon$$

Therefore  $h(\mathbf{p})$  is continuous.

STEP 3:  $h(\mathbf{p})$  is bounded away from 0.

$h(\mathbf{p})$  is the sum of non-negative terms plus one. Therefore for all values of  $\mathbf{p} \in \Delta$

---

<sup>1</sup>To see this,  $\|\mathbf{p}' - \mathbf{p}\| < \tilde{\delta}_\ell$  is the same as  $\sqrt{\sum_{k=1}^L (p'_k - p_k)^2} < \tilde{\delta}_\ell$ . Squaring both sides gives  $\sum_{k=1}^L (p'_k - p_k)^2 < \tilde{\delta}_\ell^2$ .

The left-hand side is the sum of squared (positive) terms. Thus each individual  $(p'_k - p_k)^2$  must be smaller than  $\tilde{\delta}_\ell^2$  for all  $k$ . So  $|p'_k - p_k| < \tilde{\delta}_\ell$ .

$$h(\mathbf{p}) > 0.$$

STEP 4:  $g_\ell(\mathbf{p})/h(\mathbf{p})$  is continuous.

We split this step into 2. We first show that  $\frac{1}{h(\mathbf{p})}$  is continuous at any  $\mathbf{p} \in \Delta$ . We then show that  $g_\ell(\mathbf{p}) \times \frac{1}{h(\mathbf{p})}$  is continuous (i.e. the product of two continuous functions is continuous).

By the continuity of  $h(\mathbf{p})$ , if we take  $\varepsilon = \frac{1}{2}\|h(\mathbf{p})\|$ , there exists a  $\delta_1$  such that  $\|\mathbf{p}' - \mathbf{p}\| < \delta_1$  implies that

$$\|h(\mathbf{p}') - h(\mathbf{p})\| < \frac{1}{2}\|h(\mathbf{p})\|$$

By the reverse triangle inequality:

$$\| \|h(\mathbf{p}')\| - \|h(\mathbf{p})\| \| \geq \| \|h(\mathbf{p}')\| - \|h(\mathbf{p})\| \|$$

Putting these together:

$$\| \|h(\mathbf{p}')\| - \|h(\mathbf{p})\| \| < \frac{1}{2}\|h(\mathbf{p})\|$$

So:

$$\frac{1}{2}\|h(\mathbf{p})\| < \|h(\mathbf{p}')\| < \frac{3}{2}\|h(\mathbf{p})\|$$

Therefore:

$$\frac{1}{\|h(\mathbf{p}')\|} < \frac{2}{\|h(\mathbf{p})\|}$$

By the continuity of  $h(\mathbf{p})$ , if we take  $\varepsilon = \frac{1}{2}\|h(\mathbf{p})\|^2$ , there exists a  $\delta_2$  such that  $\|\mathbf{p}' - \mathbf{p}\| < \delta_2$  implies that

$$\|h(\mathbf{p}') - h(\mathbf{p})\| < \frac{1}{2}\|h(\mathbf{p})\|^2\varepsilon$$

Take  $\delta = \min\{\delta_1, \delta_2\}$ .  $\|\mathbf{p}' - \mathbf{p}\| < \delta$  implies that

$$\left\| \frac{1}{h(\mathbf{p}')} - \frac{1}{h(\mathbf{p})} \right\| = \left\| \frac{h(\mathbf{p}) - h(\mathbf{p}')}{h(\mathbf{p})h(\mathbf{p}')} \right\| = \frac{\|h(\mathbf{p}) - h(\mathbf{p}')\|}{\|h(\mathbf{p})h(\mathbf{p}')\|} = \frac{\|h(\mathbf{p}) - h(\mathbf{p}')\|}{\|h(\mathbf{p})\|\|h(\mathbf{p}')\|} < 2 \frac{\|h(\mathbf{p}) - h(\mathbf{p}')\|}{\|h(\mathbf{p})\|\|h(\mathbf{p})\|}$$

Continuing:

$$\left\| \frac{1}{h(\mathbf{p}')} - \frac{1}{h(\mathbf{p})} \right\| < 2 \frac{\overbrace{\|h(\mathbf{p}) - h(\mathbf{p}')\|}^{< \frac{1}{2}\|h(\mathbf{p})\|^2\varepsilon}}{\|h(\mathbf{p})\|^2} < \varepsilon$$

Therefore  $\frac{1}{h(\mathbf{p})}$  is continuous.

We now define  $\tilde{h}(\mathbf{p}) = \frac{1}{h}(\mathbf{p})$  which we know now to be continuous. Our final step is to

show that  $g_\ell(\mathbf{p}) \tilde{h}(\mathbf{p})$  is continuous.

We can write:

$$\begin{aligned}
& \left\| g_\ell(\mathbf{p}') \tilde{h}(\mathbf{p}') - g_\ell(\mathbf{p}) \tilde{h}(\mathbf{p}) \right\| \\
&= \left\| g_\ell(\mathbf{p}') \tilde{h}(\mathbf{p}') - g_\ell(\mathbf{p}') \tilde{h}(\mathbf{p}) + g_\ell(\mathbf{p}') \tilde{h}(\mathbf{p}) - g_\ell(\mathbf{p}) \tilde{h}(\mathbf{p}) \right\| \\
&= \left\| g_\ell(\mathbf{p}') \left[ \tilde{h}(\mathbf{p}') - \tilde{h}(\mathbf{p}) \right] + \tilde{h}(\mathbf{p}) \left[ g_\ell(\mathbf{p}') - g_\ell(\mathbf{p}) \right] \right\| \\
&\leq \left\| g_\ell(\mathbf{p}') \right\| \left\| \tilde{h}(\mathbf{p}') - \tilde{h}(\mathbf{p}) \right\| + \left\| \tilde{h}(\mathbf{p}) \right\| \left\| g_\ell(\mathbf{p}') - g_\ell(\mathbf{p}) \right\|
\end{aligned}$$

By the continuity of  $g_\ell(\mathbf{p})$ , there exists a  $\delta_1$  such that  $\|\mathbf{p}' - \mathbf{p}\| < \delta_1$  implies that  $\|g_\ell(\mathbf{p}') - g_\ell(\mathbf{p})\| < 1$ . Then:

$$\|g_\ell(\mathbf{p}')\| = \|g_\ell(\mathbf{p}') - g_\ell(\mathbf{p}) + g_\ell(\mathbf{p})\| \leq \|g_\ell(\mathbf{p})\| + \underbrace{\|g_\ell(\mathbf{p}') - g_\ell(\mathbf{p})\|}_{<1}$$

So  $\|g_\ell(\mathbf{p}')\| < 1 + \|g_\ell(\mathbf{p})\|$ .

By the continuity of  $\tilde{h}(\mathbf{p})$ , there exists a  $\delta_2$  such that  $\|\mathbf{p}' - \mathbf{p}\| < \delta_2$  implies that  $\|\tilde{h}(\mathbf{p}') - \tilde{h}(\mathbf{p})\| < \frac{\varepsilon}{2} [1 + \|g_\ell(\mathbf{p})\|]^{-1}$ .

By the continuity of  $g_\ell(\mathbf{p})$ , there exists a  $\delta_3$  such that  $\|\mathbf{p}' - \mathbf{p}\| < \delta_3$  implies that  $\|g_\ell(\mathbf{p}') - g_\ell(\mathbf{p})\| < \frac{\varepsilon}{2} \left[ \|\tilde{h}(\mathbf{p})\| \right]^{-1}$ .

Take  $\delta = \min \{\delta_1, \delta_2, \delta_3\}$ . Then  $\|\mathbf{p}' - \mathbf{p}\| < \delta$  implies that

$$\begin{aligned}
& \left\| g_\ell(\mathbf{p}') \tilde{h}(\mathbf{p}') - g_\ell(\mathbf{p}) \tilde{h}(\mathbf{p}) \right\| \\
&\leq \left[ 1 + \left\| g_\ell(\mathbf{p}) \right\| \right] \underbrace{\left\| \tilde{h}(\mathbf{p}') - \tilde{h}(\mathbf{p}) \right\|}_{< \frac{\varepsilon}{2} [1 + \|g_\ell(\mathbf{p})\|]^{-1}} + \underbrace{\left\| \tilde{h}(\mathbf{p}) \right\| \left\| g_\ell(\mathbf{p}') - g_\ell(\mathbf{p}) \right\|}_{< \frac{\varepsilon}{2} \left[ \|\tilde{h}(\mathbf{p})\| \right]^{-1}} \\
&< \varepsilon
\end{aligned}$$

Therefore  $g_\ell(\mathbf{p}) \tilde{h}(\mathbf{p})$  is continuous, which implies that  $\frac{g_\ell(\mathbf{p})}{\tilde{h}(\mathbf{p})}$  is continuous.

□