Equilibrium Welfare Properties

230333 Microeconomics 3 (CentER) – Part II
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Introduction

- In this section we will prove:
  - The *First Welfare Theorem:* The allocation from any competitive equilibrium with transfers is Pareto optimal.
  - The *Second Welfare Theorem:* For any Pareto optimal allocation, there is a price vector that can support it as an equilibrium with transfers.
- Both theorems require complete markets, rational and locally nonsatiated preferences, and nonempty and closed production sets.
- However, the second welfare theorem requires a number of additional assumptions.

Price Equilibrium with Transfers

Definition

Given an economy specified by $\left(\{(X_i, \succeq_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \bar{\boldsymbol{\omega}}\right)$, an allocation $(\boldsymbol{x}^\star, \boldsymbol{y}^\star)$ and a price vector $\boldsymbol{p}$ constitute a *price equilibrium with transfers* if there is an assignment of wealth levels $(w_1, \ldots, w_I)$ with $\sum_{i=1}^{I} w_i = \boldsymbol{p} \cdot \bar{\boldsymbol{\omega}} + \sum_{j=1}^{J} \boldsymbol{p} \cdot \boldsymbol{y}_j^\star$ such that

(i) For every j, $\boldsymbol{y}_j^\star$ maximizes profits in Y_j; that is,

$$\boldsymbol{p} \cdot \boldsymbol{y}_j \leq \boldsymbol{p} \cdot \boldsymbol{y}_j^\star \text{ for all } \boldsymbol{y}_j \in Y_j$$

(ii) For every i, $\boldsymbol{x}_i^\star$ is maximal for $\succeq_i$ in the budget set:

$$\{\boldsymbol{x}_i \in X_i : \boldsymbol{p} \cdot \boldsymbol{x}_i \leq w_i\}$$

(iii) $\sum_{i=1}^{I} \boldsymbol{x}_i^\star = \bar{\boldsymbol{\omega}} + \sum_{j=1}^{J} \boldsymbol{y}_j^\star$.

If $w_i = \boldsymbol{p} \cdot \boldsymbol{\omega}_i + \sum_{j=1}^{J} \theta_{ij} \boldsymbol{p} \cdot \boldsymbol{y}_j \ \forall i$, then there are no transfers.

The First Fundamental Theorem of Welfare Economics

Theorem

If preferences are locally nonsatiated, and if $(\boldsymbol{x}^\star, \boldsymbol{y}^\star, \boldsymbol{p})$ is a price equilibrium with transfers, then the allocation $(\boldsymbol{x}^\star, \boldsymbol{y}^\star)$ is Pareto optimal.

Proof:

1. Because $(\boldsymbol{x}^\star, \boldsymbol{y}^\star, \boldsymbol{p})$ is an equilibrium, if $\boldsymbol{x}_i \succ_i \boldsymbol{x}_i^\star$, then $\boldsymbol{p} \cdot \boldsymbol{x}_i > w_i$.
2. Furthermore, if $\boldsymbol{x}_i \succeq_i \boldsymbol{x}_i^\star$, then $\boldsymbol{p} \cdot \boldsymbol{x}_i \geq w_i$.
    - Suppose there is an $\boldsymbol{x}_i'$ satisfying $\boldsymbol{x}_i' \succeq_i \boldsymbol{x}_i^\star$ but $\boldsymbol{p} \cdot \boldsymbol{x}_i' < w_i$.
    - By LNS, $\exists \boldsymbol{x}_i''$ arbitrarily close to $\boldsymbol{x}_i'$ where $\boldsymbol{x}_i'' \succ_i \boldsymbol{x}_i'$ and $\boldsymbol{p} \cdot \boldsymbol{x}_i'' < w_i$.
    - But this contradicts that $\boldsymbol{x}_i^\star$ was maximal in i's budget set, because by transitivity $\boldsymbol{x}_i'' \succ_i \boldsymbol{x}_i^\star$.

First Welfare Theorem Proof

3. Suppose $\exists\,(\boldsymbol{x}', \boldsymbol{y}')$ that Pareto dominates $(\boldsymbol{x}^\star, \boldsymbol{y}^\star)$.
   - By (1) & (2), $\boldsymbol{p} \cdot \boldsymbol{x}'_i \geq w_i$ $\forall$i and $\boldsymbol{p} \cdot \boldsymbol{x}'_i > w_i$ for at least one i.
   - So $\sum_{i=1}^{I} \boldsymbol{p} \cdot \boldsymbol{x}'_i > \sum_{i=1}^{I} w_i = \boldsymbol{p} \cdot \bar{\boldsymbol{\omega}} + \sum_{j=1}^{J} \boldsymbol{p} \cdot \boldsymbol{y}^\star_j$.
4. Because $\boldsymbol{y}^\star_j$ is profit-maximizing at $\boldsymbol{p}$, for all j we have $\boldsymbol{p} \cdot \boldsymbol{y}^\star_j \geq \boldsymbol{p} \cdot \boldsymbol{y}_j \; \forall \boldsymbol{y}_j \in Y_j$.
   - Therefore $\boldsymbol{p} \cdot \bar{\boldsymbol{\omega}} + \sum_{j=1}^{J} \boldsymbol{p} \cdot \boldsymbol{y}^\star_j \geq \boldsymbol{p} \cdot \bar{\boldsymbol{\omega}} + \sum_{j=1}^{J} \boldsymbol{p} \cdot \boldsymbol{y}'_j$.
5. Because $(\boldsymbol{x}', \boldsymbol{y}')$ is Pareto improving: $\sum_{i=1}^{I} \boldsymbol{x}'_i = \bar{\boldsymbol{\omega}} + \sum_{j=1}^{J} \boldsymbol{y}'_j$.
   - This implies $\sum_{i=1}^{I} \boldsymbol{p} \cdot \boldsymbol{x}'_i = \boldsymbol{p} \cdot \bar{\boldsymbol{\omega}} + \sum_{j=1}^{J} \boldsymbol{p} \cdot \boldsymbol{y}'_j$
6. But (3) & (4) imply $\sum_{i=1}^{I} \boldsymbol{p} \cdot \boldsymbol{x}'_i > \boldsymbol{p} \cdot \bar{\boldsymbol{\omega}} + \sum_{j=1}^{J} \boldsymbol{p} \cdot \boldsymbol{y}'_j$.
   - But this contradicts (5).

Separating and Supporting Hyperplane Theorems

We will use these two theorems to prove certain propositions:

Theorem (Separating Hyperplane Theorem)

Suppose that the convex sets $\mathcal{A} \subset \mathbb{R}^N$ and $\mathcal{B} \subset \mathbb{R}^N$ are disjoint. Then there is $\boldsymbol{p} \in \mathbb{R}^N$ with $\boldsymbol{p} \neq \mathbf{0}$ and a value $c \in \mathbb{R}$ such that $\boldsymbol{p} \cdot \boldsymbol{x} \geq c$ for every x in $\mathcal{A}$ and $\boldsymbol{p} \cdot \boldsymbol{y} \leq c$ for every $\boldsymbol{y} \in \mathcal{B}$.

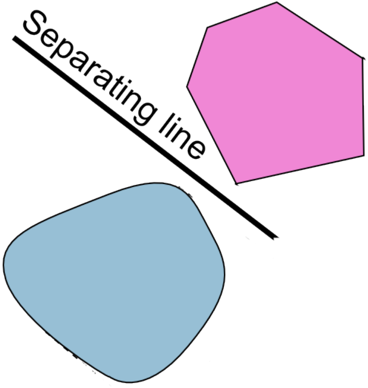
- There is a hyperplane that separates A and B, with A and B on different sides of it.

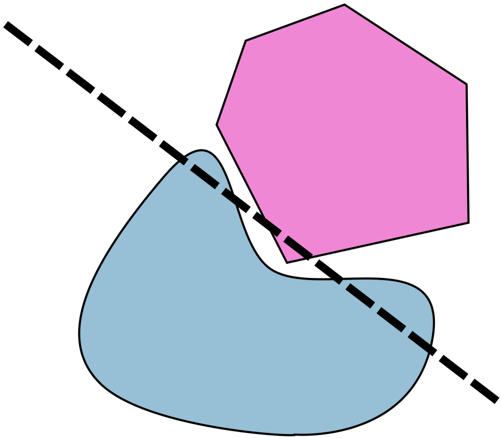
Theorem (Supporting Hyperplane Theorem)

Suppose that $\mathcal{B} \subset \mathbb{R}^N$ is convex and that $\boldsymbol{x}$ is not an element of the interior of the set $\mathcal{B}$. Then there is a $\boldsymbol{p} \in \mathbb{R}^N$ with $\boldsymbol{p} \neq \mathbf{0}$ such that $\boldsymbol{p} \cdot \boldsymbol{x} \geq \boldsymbol{p} \cdot \boldsymbol{y}$ for every $\boldsymbol{y} \in \mathcal{B}$.

Examples

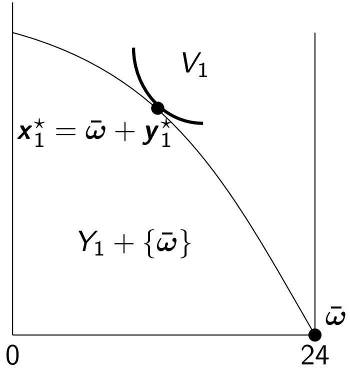
- Example 1: 2 convex, disjoint sets. SHT can be applied.
- Example 2: 2 nonconvex, disjoint sets. SHT can't be applied.





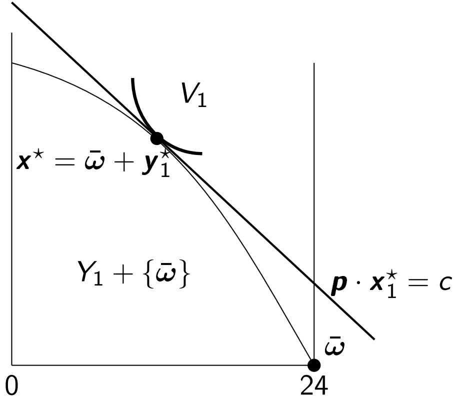
SHT Example in the Robinson Crusoe Economy

- Suppose $(\boldsymbol{x}_1^\star, \boldsymbol{y}_1^\star)$ is Pareto optimal.
- Crusoe's "better than set" is $V_1 = \{\boldsymbol{x}_1 \in X_1 : \boldsymbol{x}_1 \succ_1 \boldsymbol{x}_1^\star\}$.
- The two sets V_1 and $Y_1 + \{\bar{\boldsymbol{\omega}}\}$ are:
  - disjoint (by Pareto the optimality of $(\boldsymbol{x}_1^\star, \boldsymbol{y}_1^\star)$), and
  - convex (if $\succeq_1$ and Y_1 are convex).
- The separating hyperplane theorem can be applied.



SHT Example in the Robinson Crusoe Economy

- The SHT says $\exists \boldsymbol{p} \neq \mathbf{0}$ and a c such that $\boldsymbol{p} \cdot \boldsymbol{x}_1 \geq c \ \forall \boldsymbol{x}_1 \in V_1$ and $\boldsymbol{p} \cdot (\boldsymbol{y}_1 + \bar{\boldsymbol{\omega}}) \leq c \ \forall \boldsymbol{y}_1 + \bar{\boldsymbol{\omega}} \in Y_1 + \{\bar{\boldsymbol{\omega}}\}$.



- What we will show: if we transfer wealth $w_1 = c = \boldsymbol{p} \cdot \boldsymbol{x}_1^\star$ to Crusoe, $(\boldsymbol{x}_1^\star, \boldsymbol{y}_1^\star, \boldsymbol{p})$ is an equilibrium.

The Second Fundamental Theorem of Welfare Economics

Theorem

Consider an economy specified by $\left(\{(X_i, \succeq_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \bar{\boldsymbol{\omega}}\right)$, *and suppose that*

- *Every* X_i *is convex with* $\mathbf{0} \in X_i$.
- *Every preference relation* $\succeq_i$ *is convex, continuous and locally nonsatiated.*
- *Every* Y_j *is convex and exhibits free disposal.*

If $(\boldsymbol{x}^\star, \boldsymbol{y}^\star)$ *is a Pareto optimal allocation, where* $\boldsymbol{x}_i^\star \gg \mathbf{0}$ *for all* i, *there exists a price vector* $\boldsymbol{p} \geq \mathbf{0}$, $\boldsymbol{p} \neq \mathbf{0}$ *such that* $(\boldsymbol{x}^\star, \boldsymbol{y}^\star, \boldsymbol{p})$ *is a price equilibrium with transfers.*

Thus, there is a price vector and an assignment of wealth levels $(w_1, \ldots, w_I)$ satisfying $\sum_{i=1}^{I} w_i = \boldsymbol{p} \cdot \bar{\boldsymbol{\omega}} + \sum_{j=1}^{J} \boldsymbol{p} \cdot \boldsymbol{y}_j^\star$ such that $(\boldsymbol{x}^\star, \boldsymbol{y}^\star, \boldsymbol{p})$ is a Walrasian equilibrium.

Second Welfare Theorem Proof: Preliminaries

The goal is to show that the wealth levels $w_i = \boldsymbol{p} \cdot \boldsymbol{x}_i^\star$ for all i support $(\boldsymbol{x}^\star, \boldsymbol{y}^\star, \boldsymbol{p})$ as a price equilibrium with transfers.

Define the sets:

- $V_i = \{\boldsymbol{x}_i \in X_i : \boldsymbol{x}_i \succ_i \boldsymbol{x}_i^\star\} \subset \mathbb{R}^L$
- $V = \sum_{i=1}^I V_i = \left\{\sum_{i=1}^I \boldsymbol{x}_i \in \mathbb{R}^L : \boldsymbol{x}_1 \in V_1, \ldots, \boldsymbol{x}_I \in V_I\right\}$
- $Y = \sum_{j=1}^J Y_j = \left\{\sum_{j=1}^J \boldsymbol{y}_j \in \mathbb{R}^L : \boldsymbol{y}_1 \in Y_1, \ldots, \boldsymbol{y}_J \in Y_J\right\}$
- V is the set of aggregate consumption bundles that *could* be split across the I individuals with each i preferring it to $\boldsymbol{x}_i^\star$.
- $Y + \{\bar{\boldsymbol{\omega}}\}$ is the set of aggregate bundles producible with the given technology and endowments.

With this, we split the proof into multiple steps.

Second Welfare Theorem Proof Outline

Step 1 Every set V_i is convex.

Step 2 The sets V and $Y + \{\bar{\boldsymbol{\omega}}\}$ are convex.

Step 3 V and $Y + \{\bar{\boldsymbol{\omega}}\}$ are disjoint.

Step 4 There is a vector $\boldsymbol{p} \geq \mathbf{0}$, $\boldsymbol{p} \neq \mathbf{0}$ and a number c such that $\boldsymbol{p} \cdot \boldsymbol{z} \geq c$ for every $\boldsymbol{z} \in V$ and $\boldsymbol{p} \cdot \boldsymbol{z} \leq c$ for every $\boldsymbol{z} \in Y + \{\bar{\boldsymbol{\omega}}\}$.

Step 5 If $\boldsymbol{x}_i \succeq_i \boldsymbol{x}_i^\star$ for every i, then $\boldsymbol{p} \cdot \left(\sum_{i=1}^I \boldsymbol{x}_i\right) \geq c$.

Step 6 $\boldsymbol{p} \cdot \left(\sum_{i=1}^I \boldsymbol{x}_i^\star\right) = \boldsymbol{p} \cdot \left(\bar{\boldsymbol{\omega}} + \sum_{j=1}^J \boldsymbol{y}_j^\star\right) = c$.

Step 7 For every j, we have $\boldsymbol{p} \cdot \boldsymbol{y}_j \leq \boldsymbol{p} \cdot \boldsymbol{y}_j^\star$ for all $\boldsymbol{y}_j \in Y_j$.

Step 8 For every i, if $\boldsymbol{x}_i \succ_i \boldsymbol{x}_i^\star$, then $\boldsymbol{p} \cdot \boldsymbol{x}_i > \boldsymbol{p} \cdot \boldsymbol{x}_i^\star$.

Step 9 Steps 7 & 8 with feasibility from the Pareto optimal allocation implies that the wealth levels $w_i = \boldsymbol{p} \cdot \boldsymbol{x}_i^\star$ for all i support $(\boldsymbol{x}^\star, \boldsymbol{y}^\star, \boldsymbol{p})$ as a price equilibrium with transfers.

Second Welfare Theorem Proof

Step 1

Every set $V_i = \{\boldsymbol{x}_i \in X_i : \boldsymbol{x}_i \succ_i \boldsymbol{x}_i^\star\}$ is convex.

- We need to show that if $\boldsymbol{x}_i \in V_i$ and $\boldsymbol{x}_i' \in V_i$, then $\boldsymbol{x}_i^\alpha = \alpha \boldsymbol{x}_i + (1-\alpha)\,\boldsymbol{x}_i' \in V_i$ for all $\alpha \in [0, 1]$.
- First, by the convexity of X_i, $\boldsymbol{x}_i^\alpha \in X_i$.
- $\boldsymbol{x}_i, \boldsymbol{x}_i' \in V_i$ means $\boldsymbol{x}_i \succ_i \boldsymbol{x}_i^\star$ and $\boldsymbol{x}_i' \succ_i \boldsymbol{x}_i^\star$.
- Suppose wlog that $\boldsymbol{x}_i \succeq_i \boldsymbol{x}_i'$.
- Because preferences are convex: $\boldsymbol{x}_i^\alpha \succeq_i \boldsymbol{x}_i'\ \forall \alpha \in [0, 1]$
- Then by transitivity $\boldsymbol{x}_i^\alpha \succ_i \boldsymbol{x}_i^\star$.
- Hence $\boldsymbol{x}_i^\alpha \in V_i$.

Second Welfare Theorem Proof

Step 2

The sets V and $Y + \{\bar{\boldsymbol{\omega}}\}$ are convex.

- The sum of convex sets is convex.
  - See note at end of slide deck for $I = 2$ case.

Step 3

V and $Y + \{\bar{\boldsymbol{\omega}}\}$ are disjoint.

- V contains all bundles that can be distributed such that everyone is strictly better off than with $\boldsymbol{x}_i^\star$.
- $Y + \{\bar{\boldsymbol{\omega}}\}$ is the set of all feasible bundles.
- If they were not disjoint, then $(\boldsymbol{x}^\star, \boldsymbol{y}^\star)$ would not be Pareto optimal.

Second Welfare Theorem Proof

Step 4

There is a vector $\boldsymbol{p} \geq \mathbf{0}$, $\boldsymbol{p} \neq \mathbf{0}$ and a number c such that $\boldsymbol{p} \cdot \boldsymbol{z} \geq c$ for every $\boldsymbol{z} \in V$ and $\boldsymbol{p} \cdot \boldsymbol{z} \leq c$ for every $\boldsymbol{z} \in Y + \{\bar{\boldsymbol{\omega}}\}$.

- That such a $\boldsymbol{p} \in \mathbb{R}^L$, $\boldsymbol{p} \neq \mathbf{0}$ exists follows directly from the separating hyperplane theorem (two disjoint convex sets).
- We only need to rule out the possiblity of $p_\ell < 0$ for any ℓ.
- Because firms have free disposal, if $p_\ell < 0$ then $\boldsymbol{p} \cdot \boldsymbol{y}_j$ could become unboundedly large, violating $\boldsymbol{p} \cdot \boldsymbol{z} \leq c$ for all $\boldsymbol{z} \in Y + \{\bar{\boldsymbol{\omega}}\}$.

Second Welfare Theorem Proof

Step 5

If $\boldsymbol{x}_i \succeq_i \boldsymbol{x}_i^\star$ for every i, then $\boldsymbol{p} \cdot \left(\sum_{i=1}^{I} \boldsymbol{x}_i\right) \geq c$.

- Take $\boldsymbol{x}_i \succeq_i \boldsymbol{x}_i^\star$. By LNS we have, $\forall \varepsilon > 0$, $\exists \hat{\boldsymbol{x}}_i$ satisfying $\|\hat{\boldsymbol{x}}_i - \boldsymbol{x}_i\| \leq \varepsilon$ such that $\hat{\boldsymbol{x}}_i \succ_i \boldsymbol{x}_i$.
- By transitivity $\hat{\boldsymbol{x}}_i \succ_i \boldsymbol{x}_i^\star$ so $\hat{\boldsymbol{x}}_i \in V_i$.
- Such a $\hat{\boldsymbol{x}}_i$ exists for every consumer, so $\sum_{i=1}^{I} \hat{\boldsymbol{x}}_i \in V$.
- By Step 4: $\boldsymbol{p} \cdot \left(\sum_{i=1}^{I} \hat{\boldsymbol{x}}_i\right) \geq c$.
- As $\varepsilon \to 0$ (so $\hat{\boldsymbol{x}}_i \to \boldsymbol{x}_i \ \forall i$), we have $\boldsymbol{p} \cdot \left(\sum_{i=1}^{I} \boldsymbol{x}_i\right) \geq c$.
  - Limits preserve inequalities.

Second Welfare Theorem Proof

- As a consequence of Step 5, because $\boldsymbol{x}_i^\star \succeq_i \boldsymbol{x}_i^\star$, we have $\boldsymbol{p} \cdot \left(\sum_{i=1}^I \boldsymbol{x}_i^\star\right) \geq c$

Step 6

$$\boldsymbol{p} \cdot \left(\sum_{i=1}^I \boldsymbol{x}_i^\star\right) = \boldsymbol{p} \cdot \left(\bar{\boldsymbol{\omega}} + \sum_{j=1}^J \boldsymbol{y}_j^\star\right) = c.$$

- By feasibility, $\sum_{i=1}^I \boldsymbol{x}_i^\star = \sum_{j=1}^J \boldsymbol{y}_j^\star + \bar{\boldsymbol{\omega}} \in Y + \{\bar{\boldsymbol{\omega}}\}$.
- Therefore $\boldsymbol{p} \cdot \left(\sum_{i=1}^I \boldsymbol{x}_i^\star\right) \leq c$ because $\boldsymbol{p} \cdot \boldsymbol{z} \leq c$ for every $\boldsymbol{z} \in Y + \{\bar{\boldsymbol{\omega}}\}$.
- But Step 5 implies that $\boldsymbol{p} \cdot \left(\sum_{i=1}^I \boldsymbol{x}_i^\star\right) \geq c$
- Therefore $\boldsymbol{p} \cdot \left(\sum_{i=1}^I \boldsymbol{x}_i^\star\right) = c$.

Second Welfare Theorem Proof

Step 7

For every j, we have $\boldsymbol{p} \cdot \boldsymbol{y}_j \leq \boldsymbol{p} \cdot \boldsymbol{y}_j^\star$ for all $\boldsymbol{y}_j \in Y_j$.

- For all firms, $\forall \boldsymbol{y}_j \in Y_j$ we have $\boldsymbol{y}_j + \sum_{h \neq j} \boldsymbol{y}_h^\star \in Y$.
- From Steps 4 and 6, $\forall \boldsymbol{y}_j \in Y_j$:

$$\boldsymbol{p} \cdot \left(\bar{\boldsymbol{\omega}} + \boldsymbol{y}_j + \sum_{h \neq j} \boldsymbol{y}_h^\star \right) \leq c = \boldsymbol{p} \cdot \left(\bar{\boldsymbol{\omega}} + \boldsymbol{y}_j^\star + \sum_{h \neq j} \boldsymbol{y}_h^\star \right)$$

- Cancelling terms yields $\boldsymbol{p} \cdot \boldsymbol{y}_j \leq \boldsymbol{p} \cdot \boldsymbol{y}_j^\star$ for all $\boldsymbol{y}_j \in Y_j$, for all j.

Second Welfare Theorem Proof

Step 8

For every i, if $\boldsymbol{x}_i \succ_i \boldsymbol{x}_i^\star$, then $\boldsymbol{p} \cdot \boldsymbol{x}_i > \boldsymbol{p} \cdot \boldsymbol{x}_i^\star$.

- If $\boldsymbol{x}_i \succ_i \boldsymbol{x}_i^\star$, then $\boldsymbol{x}_i \in V_i$. From Steps 5 and 6 above we have:

$$\boldsymbol{p} \cdot \left(\boldsymbol{x}_i + \sum_{k \neq i} \boldsymbol{x}_k^\star \right) \geq c = \boldsymbol{p} \cdot \left(\boldsymbol{x}_i^\star + \sum_{k \neq i} \boldsymbol{x}_k^\star \right)$$

- Cancelling terms yields $\boldsymbol{p} \cdot \boldsymbol{x}_i \geq \boldsymbol{p} \cdot \boldsymbol{x}_i^\star$.
- Now we just need to rule out the $\boldsymbol{p} \cdot \boldsymbol{x}_i = \boldsymbol{p} \cdot \boldsymbol{x}_i^\star$ case.

Second Welfare Theorem Proof

- Suppose toward a contradition there is a $\boldsymbol{x}_i' \in \mathbb{R}_+^L$ satisfying $\boldsymbol{x}_i' \succ_i \boldsymbol{x}_i^\star$ such that $\boldsymbol{p} \cdot \boldsymbol{x}_i' = \boldsymbol{p} \cdot \boldsymbol{x}_i^\star$.
- Because $\mathbf{0} \in X_i$ and X_i is convex, $\alpha \boldsymbol{x}_i' + (1-\alpha)\,\mathbf{0} \in X_i$ for all $\alpha \in [0,1]$.
- Because $\boldsymbol{p} \geq \mathbf{0}$, $\boldsymbol{p} \neq \mathbf{0}$ and $\boldsymbol{x}_i^\star \gg \mathbf{0}$, we know that $\boldsymbol{p} \cdot \boldsymbol{x}_i^\star > 0$
- $\forall \alpha \in [0,1)$, $\alpha \boldsymbol{p} \cdot \boldsymbol{x}_i' + (1-\alpha)\,\boldsymbol{p} \cdot \mathbf{0} < \boldsymbol{p} \cdot \boldsymbol{x}_i^\star$.
- By continuity, for α close enough to 1, $\alpha \boldsymbol{x}_i' \succ_i \boldsymbol{x}_i^\star$.
- As we have found a bundle that is preferred to $\boldsymbol{x}_i^\star$ and is strictly cheaper, we have found a contradiction to what we found above.

Second Welfare Theorem Proof

Step 9

If we assign wealth levels $w_i = \boldsymbol{p} \cdot \boldsymbol{x}_i^\star$ to each consumer, $(\boldsymbol{x}^\star, \boldsymbol{y}^\star, \boldsymbol{p})$ is a price equilibrium with transfers.

This satisfies all the conditions for equilibrium:

- By Step 8: If $\boldsymbol{x}_i \succ_i \boldsymbol{x}_i^\star$, then $\boldsymbol{p} \cdot \boldsymbol{x}_i > w_i$, $\forall i$.
  - $\boldsymbol{x}_i^\star$ is maximal for $\succeq_i$ in the budget set.
- By Step 7: $\boldsymbol{p} \cdot \boldsymbol{y}_j \leq \boldsymbol{p} \cdot \boldsymbol{y}_j^\star$ for all $\boldsymbol{y}_j \in Y_j$, $\forall j$
  - $\boldsymbol{y}_j^\star$ maximizes profits in Y_j.
- Because $(\boldsymbol{x}^\star, \boldsymbol{y}^\star)$ is Pareto optimal, we have feasibility and hence market clearing in each good:

$$\sum_{i=1}^{I} \boldsymbol{x}_i^\star = \bar{\boldsymbol{\omega}} + \sum_{j=1}^{J} \boldsymbol{y}_j^\star$$

Utility Possibilities Set and Pareto Frontier

- Recall the utility possibility set:

$$\mathcal{U} = \left\{ (u_1, \ldots, u_I) \in \mathbb{R}^I : \exists \text{ feasible } (\boldsymbol{x}, \boldsymbol{y}) \text{ s.t. } u_i \leq u_i(\boldsymbol{x}_i) \ \forall i \right\}$$

- The Pareto frontier is:

$$\mathcal{UP} = \Big\{ (u_1, \ldots, u_I) \in \mathcal{U} : \text{there is no } (u_1', \ldots, u_I') \in \mathcal{U} \text{ such that } u_i' \geq u_i \ \forall i \text{ and } u_i' > u_i \text{ for some } i \Big\}$$

Theorem

A feasible allocation $(\boldsymbol{x}, \boldsymbol{y})$ *is a Pareto optimum if and only if* $(u_1(\boldsymbol{x}_1), \ldots, u_I(\boldsymbol{x}_I)) \in \mathcal{UP}$

Social Welfare

- Suppose we have the linear social welfare function:

$$W(u_1, \ldots, u_I) = \sum_{i=1}^{I} \lambda_i u_i$$

  where $\lambda_i \geq 0 \ \forall i$.

- The planner's problem is then:

$$\max_{\boldsymbol{u} \in \mathcal{U}} \boldsymbol{\lambda} \cdot \boldsymbol{u}$$

- The optimum of every linear social welfare function with $\boldsymbol{\lambda} \gg \mathbf{0}$ is Pareto optimal.
- If $\mathcal{U}$ is convex, every Pareto optimal allocation is the solution to the planner's problem for *some* welfare weights.

All Social Welfare Optima are Pareto Optimal

Theorem

If $\boldsymbol{u}^\star$ is a solution to the social welfare maximization problem

$$\max_{\boldsymbol{u}\in\mathcal{U}} \boldsymbol{\lambda}\cdot\boldsymbol{u}$$

with $\boldsymbol{\lambda} \gg \mathbf{0}$, then $\boldsymbol{u}^\star \in \mathcal{UP}$.

Proof: If not, there is another $\boldsymbol{u}' \in \mathcal{U}$ where $\boldsymbol{u}' \geq \boldsymbol{u}^\star$ and $\boldsymbol{u}' \neq \boldsymbol{u}^\star$. Then, since $\boldsymbol{\lambda} \gg \mathbf{0}$, we have $\boldsymbol{\lambda}\cdot\boldsymbol{u}' > \boldsymbol{\lambda}\cdot\boldsymbol{u}^\star$, contradicting that $\boldsymbol{u}^\star$ solved the planner's problem.

All Pareto Optimal Allocations are a Social Welfare Optimum

Theorem

If the set $\mathcal{U}$ is convex, then for any $\widetilde{\boldsymbol{u}} \in \mathcal{UP}$, there is a vector of welfare weights $\boldsymbol{\lambda} \geq \mathbf{0}$, $\boldsymbol{\lambda} \neq \mathbf{0}$, such that $\boldsymbol{\lambda} \cdot \widetilde{\boldsymbol{u}} \geq \boldsymbol{\lambda} \cdot \boldsymbol{u}$ for all $\boldsymbol{u} \in \mathcal{U}$.

Proof: If $\widetilde{\boldsymbol{u}} \in \mathcal{UP}$, then $\widetilde{\boldsymbol{u}} \in bd\,(\mathcal{U})$. Using the convexity of $\mathcal{U}$, by the supporting hyperplane theorem, $\exists \boldsymbol{\lambda} \neq \mathbf{0}$ such that $\boldsymbol{\lambda} \cdot \widetilde{\boldsymbol{u}} \geq \boldsymbol{\lambda} \cdot \boldsymbol{u}\ \forall \boldsymbol{u} \in \mathcal{U}$. Moreover $\boldsymbol{\lambda} \geq \mathbf{0}$ since otherwise you could choose a $u_i < 0$ large enough in absolute value to get $\boldsymbol{\lambda} \cdot \boldsymbol{u} > \boldsymbol{\lambda} \cdot \widetilde{\boldsymbol{u}}$.

When is $\mathcal{U}$ convex?

- If each X_i and Y_i is convex and each $u_i\,(\boldsymbol{x}_i)$ is concave, then $\mathcal{U}$ is convex (part of tutorial 3).

First-Order Conditions for Pareto Optimality

- Assume now $X_i = \mathbb{R}^L_+$ for all i.
- $\succeq_i$ is represented by $u_i(\boldsymbol{x}_i)$ which is twice continuously differentiable and satisfies $\nabla u_i(\boldsymbol{x}_i) \gg \mathbf{0}$ and $u_i(\mathbf{0}) = 0$.
- Firm j's production set is $Y_j = \left\{ \boldsymbol{y} \in \mathbb{R}^L : F_j(\boldsymbol{y}) \leq 0 \right\}$, where $F_j : \mathbb{R}^L \to \mathbb{R}$ is twice continuously differentiable, $F_j(\mathbf{0}) \leq 0$ and $\nabla F_j(\boldsymbol{y}_j) \gg \mathbf{0}$.
- $(\boldsymbol{x}, \boldsymbol{y})$ is Pareto optimal if it solves:

$$\max_{\left(\boldsymbol{x} \in \mathbb{R}^L_+, \boldsymbol{y} \in \mathbb{R}^L\right)} u_1(\boldsymbol{x}_1)$$

subject to:

- $u_i(\boldsymbol{x}_i) \geq \bar{u}_i$ for all $i = 2, \ldots, I$.
- $F_j(\boldsymbol{y}_j) \leq 0$ for all $j = 1, \ldots, J$
- $\sum_{i=1}^I x_{\ell i} \leq \bar{\omega}_\ell + \sum_{j=1}^J y_{\ell j}$ for all $\ell = 1, \ldots, L$.

First-Order Conditions for Pareto Optimality

The Lagrangian is:

$$\mathcal{L}(\cdot) = u_1(\boldsymbol{x}_1) + \sum_{i=2}^{I} \delta_i \left(u_i(\boldsymbol{x}_i) - \bar{u}_i\right) + \sum_{i=1}^{I}\sum_{\ell=1}^{L} \xi_{\ell i} x_{\ell i} -$$
$$\sum_{j=1}^{J} \gamma_j F_j(\boldsymbol{y}_j) + \sum_{\ell=1}^{L} \mu_\ell \left(\bar{\omega}_\ell + \sum_{j=1}^{J} y_{\ell j} - \sum_{i=1}^{I} x_{\ell i}\right)$$

- All constraints except for nonnegativity (with multipliers $\xi_{\ell i}$) will necessarily bind at the optimum.
- The first-order conditions are (where $\delta_1 = 1$):

$$x_{\ell i} : \delta_i \frac{\partial u_i}{\partial x_{\ell i}} + \xi_{\ell i} - \mu_\ell = 0 \text{ for all } i, \ell \text{ where } \xi_{\ell i} = 0 \text{ if } x_{\ell i} > 0$$
$$y_{\ell j} : \mu_\ell - \gamma_j \frac{\partial F_j}{\partial y_\ell} = 0 \text{ for all } j, \ell$$

First-Order Conditions for Pareto Optimality

At an interior solution $\boldsymbol{x}_i \gg \mathbf{0}$ for all i:

Equal $MRS_{i\ell\ell'}$ across i: $$\frac{\frac{\partial u_i}{\partial x_{\ell i}}}{\frac{\partial u_i}{\partial x_{\ell' i}}} = \frac{\frac{\partial u_{i'}}{\partial x_{\ell i'}}}{\frac{\partial u_{i'}}{\partial x_{\ell' i'}}} \qquad \text{for all } i, i', \ell, \ell'$$

Equal $MRTS_{j\ell\ell'}$ across j: $$\frac{\frac{\partial F_j}{\partial y_{\ell j}}}{\frac{\partial F_j}{\partial y_{\ell' j}}} = \frac{\frac{\partial F_{j'}}{\partial y_{\ell j'}}}{\frac{\partial F_{j'}}{\partial y_{\ell' j'}}} \qquad \text{for all } j, j', \ell, \ell'$$

$MRS_{i\ell\ell'} = MRTS_{j\ell\ell'}$ for each i, j: $$\frac{\frac{\partial u_i}{\partial x_{\ell i}}}{\frac{\partial u_i}{\partial x_{\ell' i}}} = \frac{\frac{\partial F_j}{\partial y_{\ell j}}}{\frac{\partial F_j}{\partial y_{\ell' j}}} \qquad \text{for all } i, j, \ell, \ell'$$

Note: If V_1 and V_2 are convex, $V = V_1 + V_2$ is convex

- Take $\boldsymbol{x}' = \boldsymbol{x}_1' + \boldsymbol{x}_2' \in V$ and and $\boldsymbol{x}'' = \boldsymbol{x}_1'' + \boldsymbol{x}_2'' \in V$.
- WTS: $\forall \alpha \in [0, 1]$ that $\alpha \boldsymbol{x}' + (1 - \alpha) \boldsymbol{x}'' \in V$.
- $\boldsymbol{x}_1' \in V_1$ and $\boldsymbol{x}_1'' \in V_1$ and similarly for $\boldsymbol{x}_2'$ and $\boldsymbol{x}_2''$.
- Because V_1 and V_2 are convex, $\forall \alpha \in [0, 1]$, $\boldsymbol{x}_1^\alpha = \alpha \boldsymbol{x}_1' + (1 - \alpha) \boldsymbol{x}_1'' \in V_1$ and similarly $\boldsymbol{x}_2^\alpha \in V_2$.
- So, by the definition of V:

$$\begin{aligned}
\alpha \boldsymbol{x}' + (1 - \alpha) \boldsymbol{x}'' &= \alpha \left(\boldsymbol{x}_1' + \boldsymbol{x}_2' \right) + (1 - \alpha) \left(\boldsymbol{x}_1'' + \boldsymbol{x}_2'' \right) \\
&= \alpha \boldsymbol{x}_1' + (1 - \alpha) \boldsymbol{x}_1'' + \alpha \boldsymbol{x}_2' + (1 - \alpha) \boldsymbol{x}_2'' \\
&= \boldsymbol{x}_1^\alpha + \boldsymbol{x}_2^\alpha
\end{aligned}$$

- This is an element of V since it is the sum of two vectors which are each elements of V_1 and V_2.

Note: Limits Preserve Inequalities

- Consider the sequence $\sum_{i=1}^{I} \widehat{\boldsymbol{x}}_i \to \sum_{i=1}^{I} \boldsymbol{x}_i$ where $\boldsymbol{p} \cdot \left(\sum_{i=1}^{I} \widehat{\boldsymbol{x}}_i\right) \geq c$.
- We want to show that this inequality is preserved at the limit: $\boldsymbol{p} \cdot \left(\sum_{i=1}^{I} \boldsymbol{x}_i\right) \geq c$.
- Suppose toward a contradiction that instead $\boldsymbol{p} \cdot \left(\sum_{i=1}^{I} \boldsymbol{x}_i\right) = d < c$.
- From the definition of the limit of a function:

$$\lim_{\sum_{i=1}^{I} \widehat{\boldsymbol{x}}_i \to \sum_{i=1}^{I} \boldsymbol{x}_i} \boldsymbol{p} \cdot \left(\sum_{i=1}^{I} \widehat{\boldsymbol{x}}_i\right) = d$$

  implies that $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall \sum_{i=1}^{I} \widehat{\boldsymbol{x}}_i$, $0 < \left|\sum_{i=1}^{I} \widehat{\boldsymbol{x}}_i - \sum_{i=1}^{I} \boldsymbol{x}_i\right| < \delta$ implies that $\left|\boldsymbol{p} \cdot \left(\sum_{i=1}^{I} \widehat{\boldsymbol{x}}_i\right) - d\right| < \varepsilon$.
- This holds for all $\varepsilon > 0$. Choose $\varepsilon = c - d$. $\exists \delta > 0$ s.t. $\forall \sum_{i=1}^{I} \widehat{\boldsymbol{x}}_i$, $0 < \left|\sum_{i=1}^{I} \widehat{\boldsymbol{x}}_i - \sum_{i=1}^{I} \boldsymbol{x}_i\right| < \delta \implies \left|\boldsymbol{p} \cdot \left(\sum_{i=1}^{I} \widehat{\boldsymbol{x}}_i\right) - d\right| < \varepsilon = c - d$.
- But then:

$$-\varepsilon < \boldsymbol{p} \cdot \left(\sum_{i=1}^{I} \widehat{\boldsymbol{x}}_i\right) - d < \varepsilon = c - d \quad \implies \quad \boldsymbol{p} \cdot \left(\sum_{i=1}^{I} \widehat{\boldsymbol{x}}_i\right) < c \quad \implies \quad \text{Contradiction!}$$