Partial Equilibrium

230333 Microeconomics 3 (CentER) – Part II
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Partial Equilibrium

- In the Robinson Crusoe economy we solved for general equilibrium in the special case of $L = 2$ and $I = J = 1$.
- Before considering the fully general case, we will study equilibria in only one good.
  - This is called *partial equilibrium*.
- Such an approach is reasonable when:
  1. The good makes up a small part of individuals' budgets, so wealth effects are negligible.
  2. Prices of all other goods in the economy are unaffected by changes in demand or supply of the good.

Partial Equilibrium Setup: Consumers

- We consider the market for a single good ℓ and treat the other $L-1$ goods as a composite commodity (e.g. money).
- We assume quasilinear utility over the composite commodity and good ℓ:

$$u_i\left(m_i, x_i\right) = m_i + \phi_i\left(x_i\right)$$

  where m_i is i's consumption of the composite good and x_i is i's consumption of the good ℓ.
  - With quasilinear utility, the wealth effects for x_i are zero.
- Assume ϕ_i is bounded above and $\phi_i'\left(x_i\right) > 0$ and $\phi_i''\left(x_i\right) < 0 \ \forall x_i \geq 0$.
- Normalize $\phi_i(0) = 0$.

Partial Equilibrium Setup: Consumers

- The price of good ℓ is p and the price of the composite is 1 (the numeraire).
- Assume that there is no initial endowment of good ℓ but $\omega_{mi} > 0\ \forall i$ and $\sum_{i=1}^{I} \omega_{mi} = \bar{\omega}_m$.

Partial Equilibrium Setup: Firms

- A firm can use z_j units of the composite good to produce q_j units of good ℓ at cost $c_j\left(q_j\right)$
- $c_j' > 0$ and $c_j'' > 0$ for all $q_j \geq 0$.
- Each firm therefore has the production set:

$$Y_j = \left\{\left(-z_j, q_j\right) : q_j \geq 0 \text{ and } z_j \geq c_j\left(q_j\right)\right\}$$

- Each consumer i owns a share $\theta_{ij} \in [0, 1]$ of each firm $j = 1, \ldots, J$, entitling them to a θ_{ij} share of that firm's profits.

Consumer's Problem

- Each consumer i chooses $(m_i, x_i) \in \mathbb{R} \times \mathbb{R}_+$ to solve:

$$\max_{m_i \in \mathbb{R}, x_i \in \mathbb{R}_+} m_i + \phi_i(x_i)$$

$$\text{subject to } m_i + p x_i \leq \omega_{mi} + \sum_{j=1}^{J} \theta_{ij} \left(p q_j - c_j(q_j) \right)$$

- Note: if were to restrict $m_i \geq 0$, then demand for x_i may depend on ω_{mi}.
- $\sum_{j=1}^{J} \theta_{ij} \left(p q_j - c_j(q_j) \right)$ is sum of profits consumer i receives from all J firms.

Consumer's Problem

- Utility is strictly increasing in both goods so the budget constraint will hold with equality.
- After substituting for m_i, the problem becomes:

$$\max_{x_i \in \mathbb{R}_+} \ \omega_{mi} + \sum_{j=1}^{J} \theta_{ij} \left(pq_j - c_j \left(q_j\right)\right) - px_i + \phi_i \left(x_i\right)$$

- We still have the $x_i \geq 0$ constraint.
- Omitting constant terms, the Lagrangian is $\mathcal{L}\left(x_i, \lambda\right) = \phi_i\left(x_i\right) - px_i + \lambda x_i$.
- The KT conditions are $\phi_i'\left(x_i\right) - p + \lambda = 0$ and $\lambda x_i = 0$ (with $\lambda \geq 0$).
- Therefore:

$$\begin{cases} \phi_i'\left(x_i\right) - p \leq 0 & \text{if } x_i = 0 \\ \phi_i'\left(x_i\right) - p = 0 & \text{if } x_i > 0 \end{cases}$$

Firm's Problem

- Given price p, firm j solves:
$$\max_{q_j \geq 0} pq_j - c_j\left(q_j\right)$$
- The first-order conditions for each firm are $p \leq c_j'\left(q_j\right)$, with equality if $q_j > 0$.

Equilibrium

- To find an equilibrium we need to find an allocation and price vector that satisfy:
  - Utility maximization.
  - Profit maximization.
  - Market clearing in both goods.
- The following Lemma will require us to only need to check for market clearing for good ℓ:

Lemma

If the allocation $\left(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_I, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_J\right)$ and price vector $\boldsymbol{p} \gg \mathbf{0}$ satisfy the market clearing condition for all goods $\ell \neq k$, and if every consumer's budget constraint is satisfied with equality, so that $\boldsymbol{p} \cdot \boldsymbol{x}_i = \boldsymbol{p} \cdot \boldsymbol{\omega}_i + \sum_{j=1}^{J} \theta_{ij} \boldsymbol{p} \cdot \boldsymbol{y}_j$ for all i, then the market for good k also clears.

Proof of Lemma

Add all consumers' budget constraints and rearrange:

$$\sum_{i=1}^{I}\sum_{\ell=1}^{L} p_\ell x_{\ell i} - \sum_{i=1}^{I}\sum_{\ell=1}^{L} p_\ell \omega_{\ell i} - \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{\ell=1}^{L} \theta_{ij} p_\ell y_{\ell j} = 0$$

$$\sum_{\ell=1}^{L} p_\ell \sum_{i=1}^{I} x_{\ell i} - \sum_{\ell=1}^{L} p_\ell \bar{\omega}_\ell - \sum_{\ell=1}^{L}\sum_{j=1}^{J} p_\ell y_{\ell j} = 0$$

$$\underbrace{\sum_{\ell \neq k}^{L} p_\ell \left(\sum_{i=1}^{I} x_{\ell i} - \bar{\omega}_\ell - \sum_{j=1}^{J} y_{\ell j}\right)}_{\text{=0 by market clearing in all goods } \ell \neq k} = -p_k \underbrace{\left(\sum_{i=1}^{I} x_{ki} - \bar{\omega}_k - \sum_{j=1}^{J} y_{kj}\right)}_{\substack{\text{Must be zero since } p_k > 0 \Rightarrow \\ \text{Market clearing in good } k}}$$

Equilibrium

- Using the Lemma, the allocation $\left(x_1^\star, \ldots, x_I^\star, q_1^\star, \ldots, q_I^\star\right)$ and price $p^\star$ constitute a competitive equilibrium iff we have:
  1. $p^\star \leq c_j'\left(q_j^\star\right)$, with equality if $q_j^\star > 0$, for all $j = 1, \ldots, J$
  2. $\phi_i'\left(x_i^\star\right) \leq p^\star$, with equality if $x_i^\star > 0$, for all $i = 1, \ldots, I$
  3. $\sum_{i=1}^I x_i^\star = \sum_{j=1}^J q_j^\star$
- If $\max_i \left\{\phi_i'(0)\right\} > \min_j \left\{c_j'(0)\right\}$ we will have $\sum_{i=1}^I x_i^\star > 0$ in equilibrium
  - We will see why shortly.
  - We will assume this is the case from now on.

Demand and Aggregate Demand

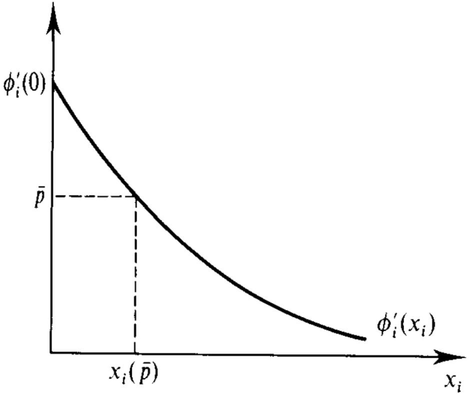
Individual Demand:

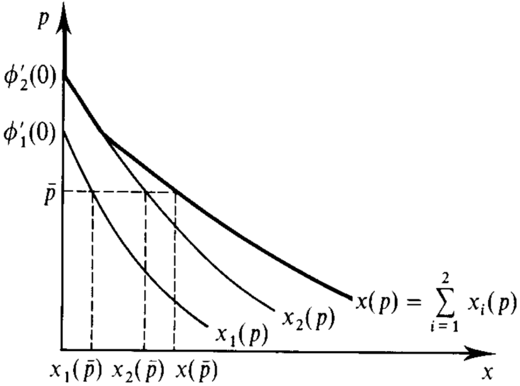
- Recall i's FOC: $\phi_i'(x_i) \leq p$, with equality if $x_i > 0$.
- Since $\phi_i' > 0$ and $\phi_i'' < 0$, ϕ_i' is positive and strictly decreasing.
- $\forall p > 0$, $\exists$ a unique x_i satisfying the FOC.
- This is $x_i(p)$, i's demand function.
  - Doesn't depend on wealth (quasilinear utility).
- $x_i(p)$ is continuous and nonincreasing in p for all $p > 0$ and is strictly decreasing for $p < \phi_i'(0)$.

Aggregate Demand:

- Aggregate demand is then $x(p) = \sum_{i=1}^{I} x_i(p)$.
- $x(p) = 0$ for all $p > \max_i \{\phi_i'(0)\}$.
- $x(p)$ is continuous and nonincreasing for $p > 0$ and strictly decreasing for all $p < \max_i \{\phi_i'(0)\}$.

Demand and Aggregate Demand





Source: Mas-Colell, A., et al. (1995) *Microeconomic Theory*

Supply and Aggregate Supply

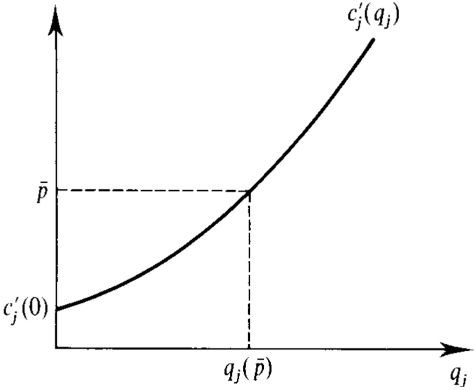
Individual Supply:

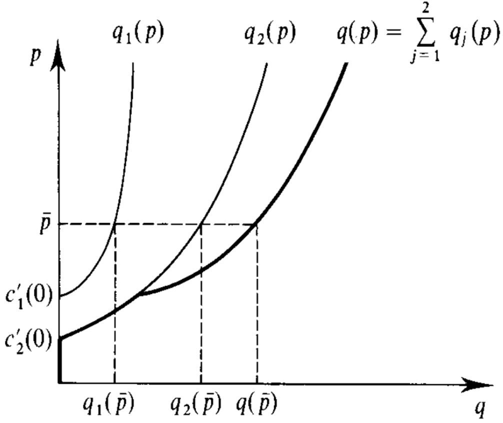
- Recall j's FOC: $c'_j\left(q_j\right) \geq p$ with equality if $q_j > 0$.
- Since $c'_j > 0$ and $c''_j > 0$, c'_j is positive and strictly increasing.
- Assume further that $c'_j\left(q_j\right) \longrightarrow \infty$ as $q_j \longrightarrow \infty$, $\forall j$.
- $\forall p > 0$, $\exists$ a unique q_j satisfying the FOC.
- This is $q_j\left(p\right)$, j's supply function.
- $q_j\left(p\right)$ is continuous and nondecreasing at all $p > 0$ and is strictly increasing at any $p > c'_j\left(0\right)$.

Aggregate Supply:

- Aggregate supply is then $q\left(p\right) = \sum_{j=1}^{J} q_j\left(p\right)$.
- $q\left(p\right) = 0$ for all $p < \min_j \left\{c'_j\left(0\right)\right\}$.
- $q\left(p\right)$ is continuous and nondecreasing at all $p > 0$ and is strictly increasing at any $p > \min_j \left\{c'_j\left(0\right)\right\}$.

Supply and Aggregate Supply





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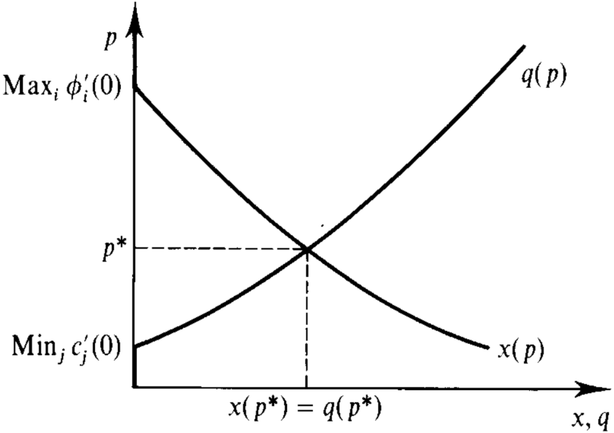
Equilibrium

- Equilibrium occurs with a $p^\star$ satisfying $x(p^\star) - q(p^\star) = 0$.
- We assume $\max_i \{\phi_i'(0)\} > \min_j \left\{c_j'(0)\right\}$.
- There cannot be an equilibrium with either $p > \max_i \{\phi_i'(0)\}$ or $p < \min_j \left\{c_j'(0)\right\}$.
- At $p = \min_j \left\{c_j'(0)\right\}$, we have $x(p) > 0$ and $q(p) = 0$ so $x(p) - q(p) > 0$.
- At $p = \max_i \{\phi'(0)\}$, we have $x(p) = 0$ and $q(p) > 0$, so $x(p) - q(p) < 0$.
- Since $x(p) - q(p)$ is continuous and strictly decreasing, the existence of a *unique* equilibrium $p^\star$ is guaranteed.

Equilibrium



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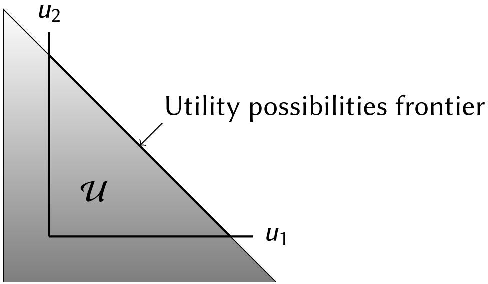
Utility Possibility Set with Quasilinear Preferences

- The *utility possibility set* for fixed $\left(\bar{x}_1, \ldots, \bar{x}_I, \bar{q}_1, \ldots, \bar{q}_J\right)$ in our quasilinear case is:

$$\mathcal{U} = \left\{(u_1, \ldots, u_I) : \sum_{i=1}^{I} u_i \leq \sum_{i=1}^{I} \phi_i\left(\bar{x}_i\right) + \bar{\omega}_m - \sum_{j=1}^{J} c_j\left(\bar{q}_j\right)\right\}$$

- The *utility possibility frontier* is the boundary of this set.
- Here, the utility possibility frontier is a hyperplane. For $I = 2$:



Utility Possibility Set with Quasilinear Preferences

- Utility can be transferred between individuals one-for-one through transfers of the numeraire.
- Changes in consumption and production levels shifts the utility possibility frontier in and out.
- When the frontier is shifted out as far as possible, the set of Pareto optimal allocations is the frontier.

Optimal Consumption and Production

- Optimal consumption and production is therefore the solution to:

$$\max_{\substack{(x_1,\ldots,x_I)\geq \mathbf{0} \\ (q_1,\ldots,q_J)\geq \mathbf{0}}} \quad \sum_{i=1}^{I} \phi_i\left(x_i\right) - \sum_{j=1}^{J} c_j\left(q_j\right) + \bar{\omega}_m$$

$$\text{subject to } \sum_{i=1}^{I} x_i - \sum_{j=1}^{J} q_j = 0$$

- The first-order conditions are (with μ being the multiplier on the constraint):
  - $\mu \leq c_j'\left(q_j^\star\right)$ with equality if $q_j^\star > 0$, for $j = 1, \ldots, J$.
  - $\phi_i'\left(x_i^\star\right) \leq \mu$ with equality if $x_i^\star > 0$, for $i = 1, \ldots, I$.
  - $\sum_{i=1}^{I} x_i^\star = \sum_{j=1}^{J} q_j^\star$.
- These are precisely the equilibrium conditions as before with μ replacing $p^\star$.

The First Fundamental Theorem of Welfare Economics

- From this example, any competitive equilibrium must be Pareto optimal because it would satisfy the FOCs when $\mu = p^\star$.
- This is the first fundamental welfare theorem in the context of a two-good quasilinear model:

Theorem

If the price $p^\star$ and allocation $\left(x_1^\star, \ldots, x_I^\star, q_1^\star, \ldots, q_J^\star\right)$ constitute a competitive equilibrium, then this allocation is Pareto optimal.

Long-Run Competitive Equilibrium

- There are an infinite number of potential firms with an identical cost function $c(q)$, where $c(0) = 0$.
- q is the individual output of a firm (will be identical across active firms in equilibrium).
- In the long run, firms exit if they can't produce any positive output without making a loss.

Long-Run Competitive Equilibrium

Definition

Given an aggregate demand function $x(p)$ and a cost function $c(q)$ for each potentially active firm having $c(0) = 0$, a triple $\left(p^\star, q^\star, J^\star\right)$ is a *long-run competitive equilibrium* if we have:

(i) Profit maximization:

$$q^\star \text{ solves } \max_{q \geq 0} \; p^\star q - c(q)$$

(ii) Market clearing:

$$x\left(p^\star\right) = J^\star q^\star$$

(iii) Free entry:

$$p^\star q^\star - c\left(q^\star\right) = 0$$

Long-Run Aggregate Supply Correspondence

- Let $Q = Jq$ be total industry output.
- The long-run aggregate supply correspondence is defined as:

$$Q(p) = \begin{cases} \infty & \text{if } \pi(p) > 0 \\ \{Q \geq 0 : Q = Jq \text{ for } J \in \mathbb{N} \cup \{0\} \text{ and } q \in q(p)\} & \text{if } \pi(p) = 0 \end{cases}$$

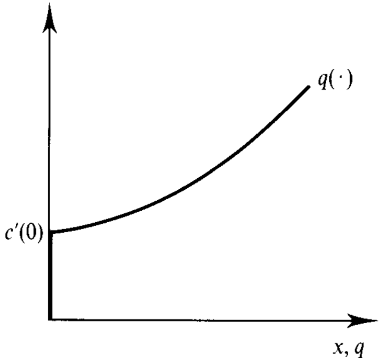
- $p^\star$ is therefore a long-run competitive equilibrium price iff $x\left(p^\star\right) \in Q\left(p^\star\right)$.

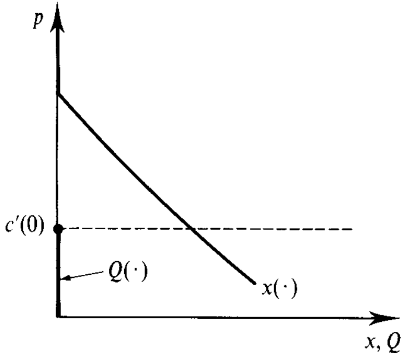
Constant Marginal Cost Example

- Suppose $c(q) = cq$ for some $c > 0$.
- Assume that $x(c) > 0$.
- If $p^\star > c$, then $Q(p) = \infty$
  $\implies$ can't be an equilibrium.
- If $p^\star < c$, then $q = 0$ for all firms, but $x(p) > 0$
  $\implies$ can't be an equilibrium.
- If $p^\star = c$, then $\pi(p) = 0$ for all $q \geq 0$
  $\implies$ Any $J^\star$ and $q^\star$ satisfying $J^\star q^\star = x(c)$ is then a long-run equilibrium
  - The number of firms is indeterminate.

Strictly Convex Costs Example

- Now assume $c(\cdot)$ is strictly convex and $x(c'(0)) > 0$.
- If $p > c'(0)$, then $\pi(p) > 0$ so $Q(p) = \infty$
  $\Longrightarrow$ can't be an equilibrium.
- If $p \leq c'(0)$, then $q = 0$ for all firms, while $x(p) > 0$
  $\Longrightarrow$ can't be an equilibrium.
- With convex costs, no long-run competitive equilibrium can exist.



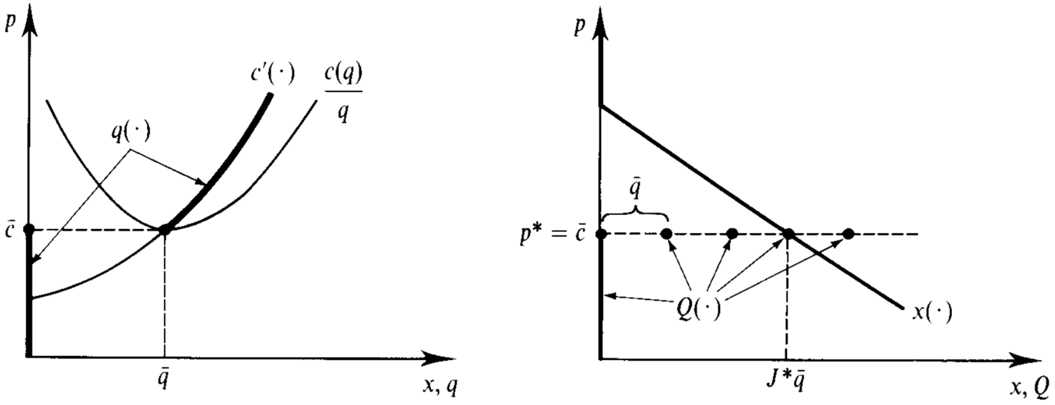


Source: Mas-Colell, A., et al. (1995) *Microeconomic Theory*

Positive Efficient Scale

- To have an equilibrium with a determinate number of firms, the long-run cost function must exhibit a *strictly positive efficient scale*.
  - There must exist a strictly positive output level $\bar{q}$ at which a firm's average costs of production are minimized.
- Let $\bar{c} = \frac{c(\bar{q})}{\bar{q}}$ be the minimum average cost, where $x(\bar{c}) > 0$.
- If $p^\star > \bar{c}$, then profits would be positive at $\bar{q}$.
- If $p^\star < \bar{c}$, then profits would be negative $\forall q > 0$.
- At $p^\star = \bar{c}$, firms optimize with $\bar{q}$.
- The equilibrium number of active firms is then $J^\star = \frac{x(\bar{c})}{\bar{q}}$.
- Note that this requires that $\frac{x(\bar{c})}{\bar{q}} \in \mathbb{N} \cup \{0\}$.

Graphical depiction with $J^\star = 3$



Source: Mas-Colell, A., et al. (1995) *Microeconomic Theory*

- If the efficient scale for one firm is large relative to the size of market demand, we may end up with situations where $J^\star = 1$ (natural monopoly).