# General Equilibrium: Introduction

230333 Microeconomics 3 (CentER) – Part II
Tilburg University

# 230333 Part II: Syllabus Summary

**Contact:** cbtwalsh@uvt.nl.
**TA:** Hazal Sezer, a.h.sezer@tilburguniversity.edu
**Topics:** 7 lectures, covering:

- Partial equilibrium and partial equilibrium with market failures (externalities/public goods).
- General equilibrium: welfare, existence, uniqueness, stability.

**Book:** Mas-Colell, Whinston and Green
**Assessment:**

- Two problem sets (each worth 5% of final grade).
- Final exam (closed book, 80%): questions from parts 1 and 2.

# Definition of an Economy

- There are $I > 0$ consumers, $J > 0$ firms and $L$ commodities in the economy.
- Each consumer $i$ is characterized by a consumption set $X_i \subset \mathbb{R}^L$ and a complete and transitive preference relation $\succeq_i$ defined over $X_i$
- Each firm $j$ is characterized by a nonempty and closed production set $Y_j \subset \mathbb{R}^L$
- There is an initial endowment $\bar{\boldsymbol{\omega}} = (\bar{\omega}_1, \dots, \bar{\omega}_L) \in \mathbb{R}^L_+$ of each good.
- An *economy* can be summarized by:

$$\mathcal{E} = \left( \{(X_i, \succeq_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \bar{\boldsymbol{\omega}} \right)$$

# Example: A Day in the Life of Robinson Crusoe

- Robinson Crusoe is a lone castaway on the Island of Despair (Defoe, 1719).
- There are two goods: coconuts and hours of leisure: $L = 2$.
- Crusoe is the only consumer on the island: $I = 1$.
- A coconut-producing firm is the only firm: $J = 1$.
- Crusoe can consume $x_{11} \geq 0$ coconuts but at most 24 hours of leisure per day:

$$X_1 = \mathbb{R}_+ \times [0, 24]$$

- Crusoe has preferences over $X_1$ given by $\succeq_1$.
- The firm's production technology is producing coconuts from leisure:

$$Y_1 = \left\{ (y_{11}, y_{21}) \in \mathbb{R}^2 : y_{11} \leq f\left(-y_{21}\right) \right\}$$

- The initial endowment is no coconuts and 24 hours of leisure:

$$\bar{\omega} = (0, 24)$$

# Feasible Allocations

**Definition**

An *allocation* $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_I, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_J)$ is a specification of a consumption vector $\boldsymbol{x}_i \in X_i$ for each consumer $i = 1, \ldots, I$ and production vector $\boldsymbol{y}_j \in Y_j$ for each firm $j = 1, \ldots, J$.

**Definition**

The allocation $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_I, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_J)$ is *feasible* if

$$\sum_{i=1}^{I} x_{\ell i} = \bar{\omega}_\ell + \sum_{j=1}^{J} y_{\ell j} \quad \forall \ell = 1, \ldots, L$$

- Note: with equality because of free disposal.

# Feasible Allocations on the Island

An allocation $(\boldsymbol{x}_1, \boldsymbol{y}_1) \in X_1 \times Y_1$ is feasible if:

$$\begin{aligned} \text{Coconuts: } x_{11} &= 0 + y_{11} \quad && \text{with } x_{11} \geq 0 \\ \text{Leisure: } x_{21} &= 24 + y_{21} \quad && \text{with } x_{21} \in [0, 24] \end{aligned}$$

- Combining feasibility with the production technology:
  - Output of coconuts is

    $$y_{11} \leq f(-y_{21}) = f\Bigg(\underbrace{24 - x_{21}}_{\text{Labor hours}}\Bigg)$$

    (with equality under no disposal).
  - Under no disposal, coconut consumption would be:

    $$x_{11} = f(24 - x_{21})$$

# Pareto Optimality

Denote the set of feasible allocations by:

$$\mathcal{A} = \left\{ (\boldsymbol{x}, \boldsymbol{y}) \in \prod_{i=1}^{I} X_i \times \prod_{j=1}^{J} Y_j : \sum_{i=1}^{I} \boldsymbol{x}_i = \bar{\boldsymbol{\omega}} + \sum_{j=1}^{J} \boldsymbol{y}_j \right\} \subset \mathbb{R}^{L(I+J)}$$

## Definition

A feasible allocation $(\boldsymbol{x}, \boldsymbol{y})$ is *Pareto optimal* if there is no other feasible allocation $(\boldsymbol{x}', \boldsymbol{y}') \in \mathcal{A}$ that *Pareto dominates* it, that is, if there is no feasible allocation $(\boldsymbol{x}', \boldsymbol{y}') \in \mathcal{A}$ such that $\boldsymbol{x}'_i \succeq_i \boldsymbol{x}_i$ for all $i$ and $\boldsymbol{x}'_i \succ_i \boldsymbol{x}_i$ for some $i$.

# Pareto Optimality on the Island of Despair

- Assume Crusoe's preferences over $X_1$ are continuous, convex and monotone and can be represented by:
$$u_1\left(x_{11}, x_{21}\right) = x_{11}x_{21}$$
- Assume the firm's production technology is:
$$Y_1 = \left\{\left(y_{11}, y_{21}\right) : 0 \leq y_{11} \leq \sqrt{-y_{21}} \text{ and } y_{21} \in [-24, 0]\right\}$$
    - $z$ hours of labor can produce $\sqrt{z}$ coconuts.

# Pareto Optimality on the Island of Despair

- Because $I = 1$ a Pareto optimal allocation $\boldsymbol{x}_1^\star$ maximizes $x_{11}x_{21}$ subject to:
  - $x_{11} \geq 0$ and $x_{21} \in [0, 24]$ (consumption set)
  - $y_{11} \leq \sqrt{-y_{21}}$ (production technology)
  - $x_{11} = y_{11}$ and $x_{21} = 24 + y_{21}$ (feasibility)
- Because utility is increasing in both goods, there is no disposal: $x_{11} = \sqrt{24 - x_{21}}$
- The problem becomes (combining all constraints):

$$\max_{x_{21} \in [0,24]} \left(\sqrt{24 - x_{21}}\right) x_{21}$$

- This is maximized at $x_{21} = 16$, which implies:

$$(x_{11}, x_{21}) = \left(\sqrt{8}, 16\right)$$

$$(y_{11}, y_{21}) = \left(\sqrt{8}, -8\right)$$

# A Private Ownership Economy

- A market exists for each good and consumers and firms are price takers. Prices are given by $\boldsymbol{p} \in \mathbb{R}^L$.
- Each consumer $i$ has an initial endowment $\boldsymbol{\omega}_i \in \mathbb{R}^L_+$, where $\bar{\omega}_\ell = \sum_{i=1}^I \omega_{\ell i} \; \forall \ell$.
- Each consumer $i$ has a claim to a share $\theta_{ij} \in [0,1]$ of the profits of firm $j$, where $\sum_{i=1}^I \theta_{ij} = 1 \; \forall j$.
- Consumer $i$'s budget set with price vector $\boldsymbol{p} \in \mathbb{R}^L$ is then:

$$\mathcal{B}_i\left(\boldsymbol{p}\right) = \left\{ \boldsymbol{x}_i \in X_i : \boldsymbol{p} \cdot \boldsymbol{x}_i \leq \boldsymbol{p} \cdot \boldsymbol{\omega}_i + \sum_{j=1}^{J} \theta_{ij} \boldsymbol{p} \cdot \boldsymbol{y}_j \right\}$$

- A *private ownership economy* can be summarized by:

$$\mathcal{E}_p = \left( \{(X_i, \succeq_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \{(\boldsymbol{\omega}_i, \theta_{i1}, \ldots, \theta_{iJ})\}_{i=1}^I \right)$$

# Robinson Crusoe's Private Ownership Economy

- There is a market for coconuts and labor.
- The price of a coconut is $p_1$ and the price of one hour of leisure is $p_2$.
- Crusoe owns the full initial endowment: $\boldsymbol{\omega}_1 = \bar{\boldsymbol{\omega}} = (0, 24)'$.
- Crusoe owns a 100% share in the coconut firm: $\theta_{11} = 1$.
- The firm hires Crusoe for labor and produces coconuts. The firm then sells Crusoe the coconuts.
- Crusoe's budget set is then all $\boldsymbol{x}_i \in X_i$ that satisfy:

$$\underbrace{p_1 x_{11} + p_2 x_{21}}_{\text{Cost of consumption}} \leq \underbrace{24 p_2}_{\text{Value of endowment}} + \underbrace{1 \times [p_1 y_{11} + p_2 y_{21}]}_{\text{Share of firm's profits}}$$

# Walrasian Equilibrium

**Definition**

Given a private ownership economy $\mathcal{E}_p$, an allocation $(\boldsymbol{x}^\star, \boldsymbol{y}^\star)$ and a price vector $\boldsymbol{p} \in \mathbb{R}^L$ constitute a *Walrasian equilibrium* if:

(i) For every $j$, $\boldsymbol{y}_j^\star$ maximizes profits in $Y_j$; that is $\boldsymbol{p} \cdot \boldsymbol{y}_j \leq \boldsymbol{p} \cdot \boldsymbol{y}_j^\star \; \forall \boldsymbol{y}_j \in Y_j$.

(ii) For every $i$, $\boldsymbol{x}_i^\star$ is maximal for $\succeq_i$ in the budget set $\mathcal{B}_i(\boldsymbol{p})$

(iii) $\sum_{i=1}^{I} \boldsymbol{x}_i^\star = \bar{\boldsymbol{\omega}} + \sum_{j=1}^{J} \boldsymbol{y}_j^\star$

# Solving for the Equilibrium: Firm's Problem

- The firm solves:

$$\max_{\{\boldsymbol{y}_1 \in \mathcal{Y}_1\}} p_1 y_{11} + p_2 y_{21}$$

- If $\boldsymbol{p} \gg \mathbf{0}$, the firm will set $y_{11} = \sqrt{-y_{21}}$ (no disposal):

$$\max_{y_{21}} p_1 \sqrt{-y_{21}} + p_2 y_{21}$$

- First-order conditions:

$$-\frac{p_1}{2\sqrt{-y_{21}}} + p_2 = 0 \quad \Longrightarrow \quad y_{21} = -\frac{p_1^2}{4p_2^2} \quad \text{and} \quad y_{11} = \frac{p_1}{2p_2}$$

- Firm's profit function is then:

$$\pi_1(\boldsymbol{p}) = \frac{p_1^2}{2p_2} - \frac{p_1^2}{4p_2} = \frac{p_1^2}{4p_2}$$

# Solving for the Equilibrium: Consumer's Problem

- The consumer solves:

$$\max_{\{x_{11}\geq 0, x_{21}\in[0,24]\}} x_{11}x_{21}$$

  subject to:

$$p_1 x_{11} + p_2 x_{21} \leq 24 p_2 + \frac{p_1^2}{4p_2}$$

- The budget constraint will bind, so:

$$x_{11} = \frac{p_2}{p_1}(24 - x_{21}) + \frac{p_1}{4p_2}$$

- The problem is then:

$$\max_{\{x_{21}\in[0,24]\}} \left(\frac{p_2}{p_1}(24 - x_{21}) + \frac{p_1}{4p_2}\right) x_{21}$$

# Solving for the Equilibrium: Consumer's Problem

- The problem again:

$$\max_{\{x_{21}\in[0,24]\}} \left( \frac{p_2}{p_1}(24 - x_{21}) + \frac{p_1}{4p_2} \right) x_{21}$$

- Taking first-order conditions (consider only interior solutions):

$$\frac{p_2}{p_1}(24 - x_{21}) + \frac{p_1}{4p_2} = \frac{p_2}{p_1} x_{21}$$

- Solving for $x_{21}$:

$$x_{21} = 12 + \frac{p_1^2}{8p_2^2}$$

- From the budget constraint, demand of coconuts is then:

$$x_{11} = \frac{p_2}{p_1}\left(24 - \left(12 + \frac{p_1^2}{8p_2^2}\right)\right) + \frac{p_1}{4p_2} = 12\frac{p_2}{p_1} - \frac{p_1}{8p_2} + \frac{p_1}{4p_2} = 12\frac{p_2}{p_1} + \frac{p_1}{8p_2}$$

# Walrasian Equilibrium

- An allocation and price vector $\boldsymbol{p}$ constitute a Walrasian equilibrium if markets clear in all goods when firms and consumers are optimizing subject to their constraints.
- In equilibrium, total demand for coconuts equals total supply:

$$\underbrace{12\frac{p_2}{p_1} + \frac{p_1}{8p_2}}_{\text{Demand}} = \underbrace{\frac{p_1}{2p_2}}_{\text{Supply}} \implies \frac{p_1}{p_2} = \sqrt{32} = 2\sqrt{8}$$

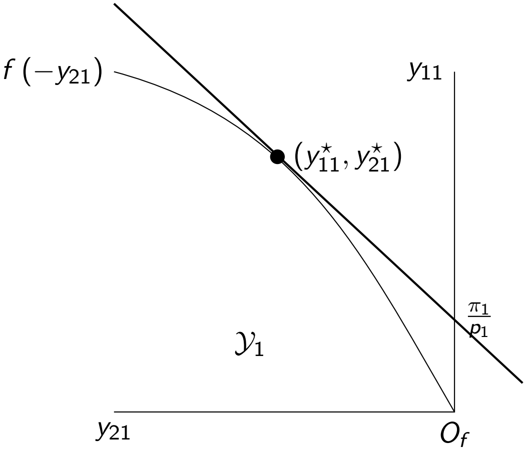
- At these prices, the market for leisure also clears:

$$x_{21} = \underbrace{24}_{\bar{\omega}_2} + y_{21} \implies \underbrace{12 + \frac{32}{8}}_{\text{Demand}} = \underbrace{24 - \frac{32}{4}}_{\text{Supply}} = 16$$

- Same as Pareto optimal outcome!
  - This is not a coincidence (1$^{\text{st}}$ Welfare Theorem).
- Any price vector where $\frac{p_1}{p_2} = 2\sqrt{8}$ is an equilibrium.
  - When normalizing prices, only 1 equilibrium.

# Graphical Analysis: Firm's Problem

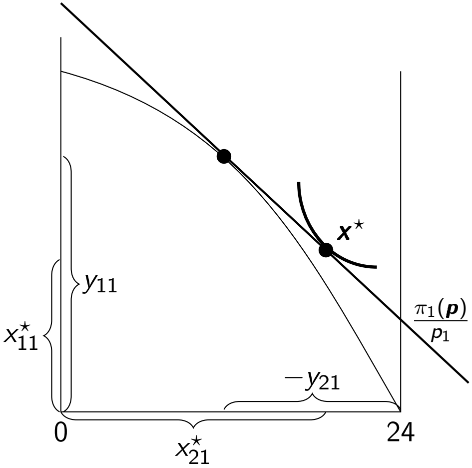
The firm's optimal choice of $y_{21}$ is when the isoprofit line $\frac{\pi_1}{p_1} - \frac{p_2}{p_1} y_{21}$ is tangent to the production set:




The isoprofit line is found by solving $\pi_1 = p_1 y_{11} + p_2 y_{21}$ for $y_{11}$.

# Graphical Analysis: Consumer's Problem

- The budget line is: $x_{11} = \frac{p_2}{p_1}(24 - x_{21}) + \frac{\pi_1(\boldsymbol{p})}{p_1}$.
- The slope is $-\frac{p_2}{p_1}$, and the intercept above $x_{21} = 24$ is $\frac{\pi_1(\boldsymbol{p})}{p_1}$, the same as the firm's isoprofit function. It therefore passes through firm's choice $(y_{11}, y_{21})$.



# Graphical Analysis: Equilibrium

- In equilibrium the slope $-\frac{p_2}{p_1}$ adjusts so that markets clear: $x_{11} = y_{11}$ and $x_{21} = 24 + y_{21}$: