Elasticities

Question 1 – Cobb-Douglas Elasticities

Your utility function for goods 1 and 2 is given by:

$$u(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}$$

(i) Derive the demand functions $x_1(p_1, p_2, m)$ and $x_2(p_1, p_2, m)$.

The *MRS* is:

$$MRS = -\frac{MU_1}{MU_2} = -\frac{\frac{1}{3}x_1^{\frac{1}{3}-1}x^{\frac{2}{3}}}{\frac{2}{3}x_1^{\frac{1}{3}}x^{\frac{2}{3}-1}} = -\frac{1}{2}\frac{x_2}{x_1}$$

The optimality condition is to set $|MRS| = \frac{p_1}{p_2}$. From this we get $x_2 = 2x_1\frac{p_1}{p_2}$. Using this expression for x_2 in the budget constraint we can solve for x_1 to get the demand (same as on previous problems):

$$x_1(p_1, p_2, m) = \frac{m}{3p_1}$$

Then using this expression for x_1 in $x_2 = 2x_1\frac{p_1}{p_2}$ we can get:

$$x_2(p_1, p_2, m) = \frac{2m}{3p_2}$$

(ii) Find the own-price elasticity of demand for good 1, ε_1.

The formula for the own-price elasticity of demand for good 1 is:

$$\varepsilon_1 = \frac{\partial x_1(p_1, p_2, m)}{\partial p_1}\frac{p_1}{x_1(p_1, p_2, m)}$$

The first term in the formula is the derivative of the demand function with respect to price. This is:

$$\frac{\partial x_1(p_1, p_2, m)}{\partial p_1} = -\frac{m}{3p_1^2}$$

Where you should recall that the derivative of $\frac{1}{x}$ is $-\frac{1}{x^2}$. We can use this in the rest of the formula:

$$\varepsilon_1 = \frac{\partial x_1(p_1, p_2, m)}{\partial p_1}\frac{p_1}{x_1(p_1, p_2, m)} = \left(-\frac{m}{3p_1^2}\right)\frac{p_1}{x_1(p_1, p_2, m)} = \left(-\frac{m}{3p_1^2}\right)\frac{p_1}{\frac{m}{3p_1}} = -1$$

Every term cancels and we are left with the own-price elasticity of demand being equal to -1.

(iii) Is good 1 an ordinary or Giffen good?

Since $\varepsilon_1 < 0$ the good is ordinary.

(iv) Find the the cross-price elasticity of good 1 with respect to good 2, ε_{12}.

Since p_2 doesn't show up in demand:

$$\varepsilon_{12} = \frac{\partial x_1(p_1, p_2, m)}{\partial p_2} \frac{p_2}{x_1(p_1, p_2, m)} = 0 \times \frac{p_2}{x_1(p_1, p_2, m)} = 0$$

(v) Are the substitutes, complements, or neither?

Since $\varepsilon_{12} = 0$ they are neither.

(vi) Find the income elasticity of demand for good 1, η_1.

The formula for the income elasticity is:

$$\eta_1 = \frac{\partial x_1(p_1, p_2, m)}{\partial m} \frac{m}{x_1(p_1, p_2, m)}$$

The first term is:

$$\frac{\partial x_1(p_1, p_2, m)}{\partial m} = \frac{1}{3p_1}$$

We can use this in the rest of the formula:

$$\eta_1 = \frac{\partial x_1(p_1, p_2, m)}{\partial m} \frac{m}{x_1(p_1, p_2, m)} = \frac{1}{3p_1} \frac{m}{x_1(p_1, p_2, m)} = \frac{1}{3p_1} \frac{m}{\frac{m}{3p_1}} = 1$$

Every term cancels and we are left with an income elasticity of 1.

(vii) Is the good normal or inferior?

Since $\eta_1 = 1$ the good is normal.

When finding the elasticities, do not just write down the answer. Show each step in the calculation.

Question 2 – Linear demand and Marginal Revenue

The market demand curve for a good is given by:

$$D(p) = 10 - p$$

(i) What is the own-price elasticity of demand at the following price and quantity pairs:

(a) $p = 4$ and $q = 6$.

(b) $p = 5$ and $q = 5$.

(c) $p = 6$ and $q = 4$.

Comment on whether demand is elastic, inelastic or unit elastic at each of these prices.

For any price, the elasticity is:

$$\varepsilon = \frac{dD(p)}{dp}\frac{p}{D(p)} = -1 \times \frac{p}{10-p} = -\frac{p}{10-p}$$

At the different prices we get:

- $\varepsilon(p=4) = -\frac{2}{3}$ (Inelastic)
- $\varepsilon(p=5) = -1$ (Unit elastic)
- $\varepsilon(p=6) = -1.5$ (Elastic)

(ii) Find the inverse demand curve, $p(q)$ for the demand curve above.

The inverse demand function is:

$$p(q) = 10 - q$$

(iii) Find the revenue function, $R(q)$ for a single firm selling to this market.

The revenue function is:

$$R(q) = p(q)\,q = (10-q)\,q = 10q - q^2$$

(iv) Find the marginal revenue function $MR(q)$.

The marginal revenue function is:

$$MR(q) = \frac{dR(q)}{dq} = 10 - 2q$$

(v) What is the marginal revenue at each of the price and quantity pairs in (i) above?

- $MR(q=6) = -2$
- $MR(q=5) = 0$
- $MR(q=4) = 2$

(vi) What does your finding in (v) tell us about the relationship between elasticity and marginal revenue?

- If demand is inelastic, marginal revenue is negative.
- If demand is elastic, marginal revenue is positive.
- If demand is unit elastic, marginal revenue is zero.

Monopoly

Question 3 – Monopoly

The (inverse) market demand for a good is given by $p(q) = 130 - q$ and there is a single producer with a cost function, $c(q) = 1600 + 10q + q^2$. Find the equilibrium price, quantity, and profits for the monopolist.

The monopolist's revenue function is:

$$R(q) = p(q)\,q = (130 - q)\,q = 130q - q^2$$

The marginal revenue function is

$$MR(q) = 130 - 2q$$

The marginal cost is

$$MC(q) = 10 + 2q$$

Setting $MR(q) = MC(q)$ and solving for q:

$$130 - 2q = 10 + 2q \quad\Longleftrightarrow\quad 4q = 120 \quad\Longleftrightarrow\quad q = 30$$

Using this in the demand curve we find that $p = 130 - 30 = 100$. The profit for the monopolist is:

$$\pi = p \times q - 1600 - 10q - q^2 = 100 \times 30 - 1600 - 10 \times 30 - 30^2 = 3000 - 1600 - 300 - 900 = 200$$

So $q = 30$, $p = 100$ and $\pi = 200$.

Question 4 – Monopoly vs. Perfect Competition with Linear Demand and Costs

The inverse demand curve for a good is given by:

$$p(q) = 10 - q$$

The cost function for the monopolist is:

$$c(q) = 2q$$

(i) Derive the optimal price an quantity the monopolist will choose.

The monopolist's revenue function is:

$$R(q) = p(q)\,q = (10 - q)\,q = 10q - q^2$$

The marginal revenue is:

$$MR(q) = 10 - 2q$$

The marginal cost is:

$$MC(q) = 2$$

To find the optimal quantity, set $MR(q) = MC(q)$:

$$10 - 2q = 2 \quad\Longleftrightarrow\quad 2q = 8 \quad\Longleftrightarrow\quad q = 4$$

The price is then found using the demand curve: $p = 10 - 4 = 6$.

(ii) Confirm that the monopolist charges a mark-up over marginal cost of $\frac{1}{1-\frac{1}{|\varepsilon|}}$. You can use your answer in Q2 (i).

The mark-up over marginal cost here is 3 (the monopolist charges 3 times the marginal cost). In Q2 (i) we found that $\varepsilon\,(p = 6) = -1.5$. So:

$$\frac{1}{1-\frac{1}{|\varepsilon|}} = \frac{1}{1-\frac{1}{1.5}} = \frac{1}{1-\frac{2}{3}} = \frac{1}{\frac{1}{3}} = 3$$

So indeed the monopolist charges a mark-up over marginal cost of $\frac{1}{1-\frac{1}{|\varepsilon|}}$.

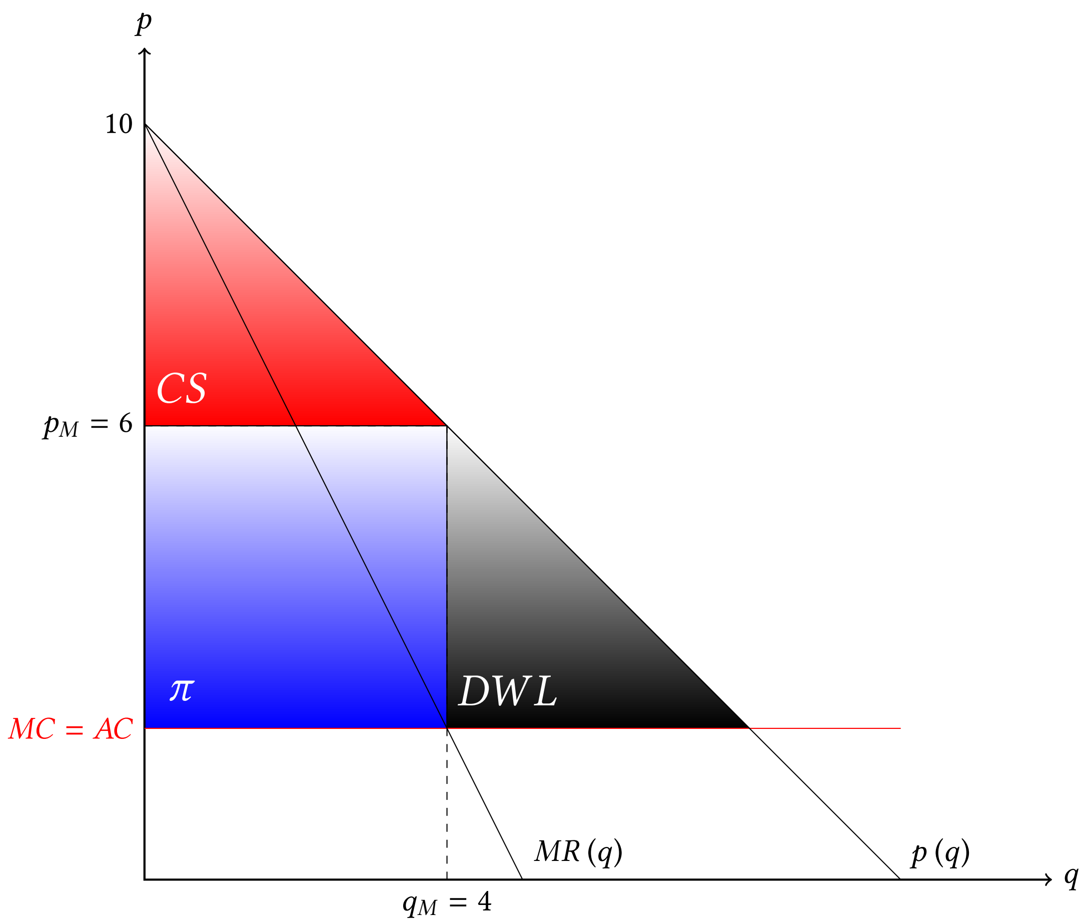
(iii) Find the profit the monopolist makes.

The profit is $\pi = (p - AC)\,q = (6-2)\times 4 = 16$.

(iv) Draw a diagram showing the following:

- The inverse demand curve.
- The marginal revenue curve.
- The marginal cost curve.
- The monopolist's price and quantity.
- The consumer surplus.
- The monopolist's profits.
- The deadweight loss.

The scaling of the diagram does not need to be exact. However, the lines/curves should have the correct shape.



(v) What is the consumer surplus?

To find the consumer surplus, we find the area of the triange:

$$CS_M = \frac{1}{2} \times (10 - 6) \times 4 = 8$$

(vi) What is the deadweight loss, if any?

To find the deadweight loss, we find the area of the triange:

$$DWL_M = \frac{1}{2} \times (6 - 2) \times (8 - 4) = 8$$

Now assume the market was instead served by a large number of **perfectly competitive** firms.

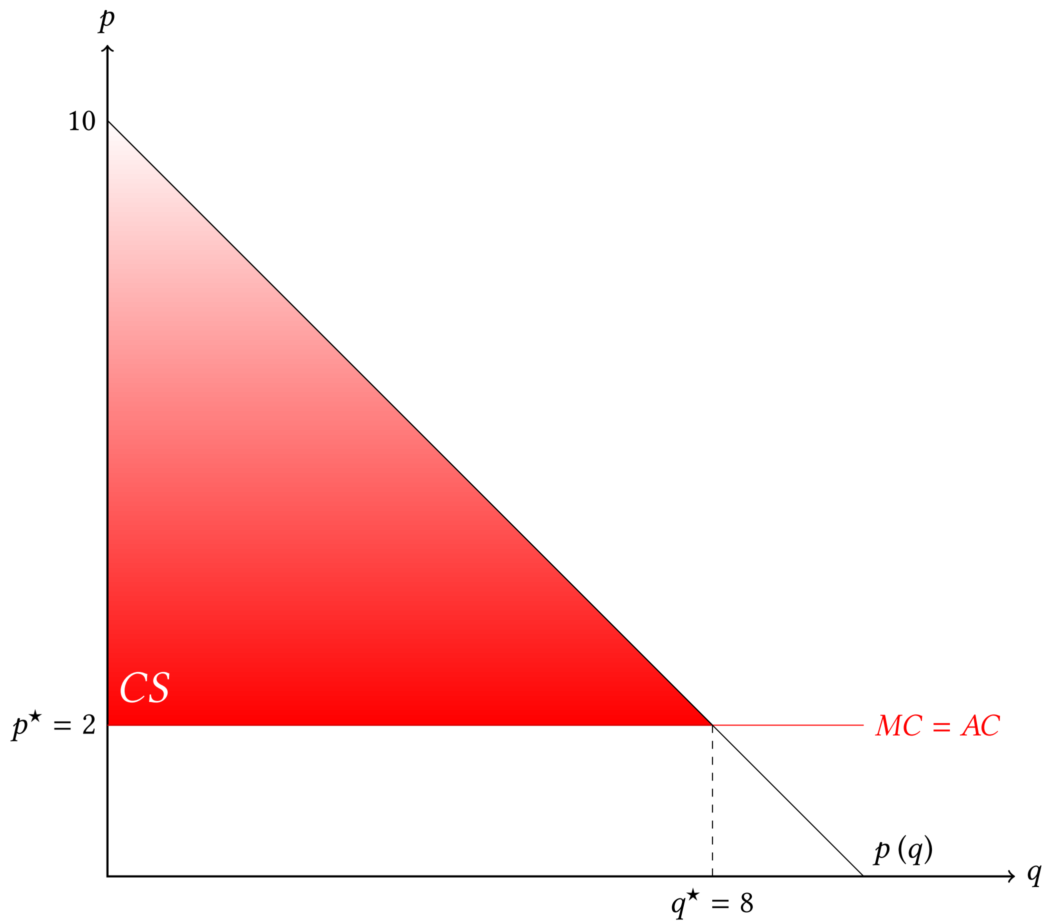
(vi) What would the price and quantity be? What profit would each of the firms make?

The quantity occurs where $p = MC$. So $p = 2$. The quantity demanded is then $q = 10 - p = 10 - 2 = 8$. Since $MC = AC$, and $p = AC$, the profits would be zero.

(vii) Draw a diagram showing the following:

- The inverse demand curve.
- The marginal cost curve.
- The perfectly competitive price and quantity.
- The consumer surplus.
- The producer surplus.

There is zero producer surplus and zero deadweight loss.



(viii) What is the consumer surplus?

The area of the triangle:

$$CS^{\star} = \frac{1}{2}(10 - 2) \times 8 = 32$$

(ix) What is the producer surplus?

$PS^{\star} = 0$

(x) What is the deadweight loss, if any?

$DWL^{\star} = 0$

(xi) How do each of price, quantity, profits, producer surplus, consumer surplus and deadweight loss compare between monopoly and perfect competition?

- The monopolist charges a higher price
- The monopolist sells a lower quantity.
- The monopolist makes positive profits and the perfectly competitive firms make zero profits.
- The monopolist gets positive producer surplus and the perfectly competitive firms get zero producer surplus.
- The consumer surplus is larger under perfect competition.
- There is a deadweight loss in monopoly. There is no deadweight loss in perfect competition.

Question 5 – Natural Monopoly

The inverse demand curve for a good is given by:

$$p(q) = 10 - q$$

The cost function for the monopolist is:

$$c(q) = 9$$

(i) Derive the optimal price an quantity the monopolist will choose.

The marginal revenue function is $10 - 2q$. The marginal cost is zero. Therefore the optimal quantity is $q = 5$ and the price is $p = 5$.

(ii) Find the profit the monopolist makes.

The profit is $p \times q - c(q) = 25 - 9 = 16$.

(iii) What is the consumer surplus?

The consumer surplus is $\frac{1}{2}(10-5) \times 5 = 12.5$

Suppose the government introduced some regulation which forced the monopolist to charge at marginal cost.

(iv) What would the new price, quantity and profits be?

Since marginal cost is zero, price is zero. The quantity demanded will then be 10. The profits will be -9 because the firm will have zero revenue.

(v) What is the consumer surplus?

The consumer surplus is $\frac{1}{2} \times 10 \times 10 = 50$.

Suppose the government introduced some regulation which forced the monopolist to charge a price such that it makes zero profits overall.

(vi) What would the price, quantity and profits be?

We set average cost equal to price: $AC(q) = p(q)$. So:

$$\begin{aligned} \frac{9}{q} &= 10 - q \\ 9 &= 10q - q^2 \\ q^2 - 10q + 9 &= 0 \\ (q-1)(q-9) &= 0 \end{aligned}$$

So $AC(q)$ intersects $p(q)$ at 1 and 9. We are trying to make consumer surplus as big as possible so we should choose $q = 9$. $q = 1$ makes profits zero but the consumer surplus will be small. The price with $q = 9$ is then $p = 1$. The profits are zero.

(vii) What is the consumer surplus?

$$CS = \frac{1}{2} \times (10 - 1) \times 9 = 40.5$$

There are three possible policies the government can pursue, the outcomes of which you have solved for above:

- No regulation.
- Forcing the monopolist to charge at marginal cost.
- Forcing the monopolist to charge a price such that it makes zero profits.

(viii) Which policy maximizes consumer surplus? The $p = MC$ policy maximizes consumer surplus ($CS = 50$).

(ix) Which policy maximizes consumer surplus *plus* profits? The sum of consumer surplus in each case is:

(a) 16 + 12.5 = 28.5

(b) 50 - 9 = 41

(c) 0 + 40.5 = 40.5

So charging $p = MC$ actually makes the largest surplus plus profits (but the monopolist will not be very happy with this arrangement).

Question 6 – Two-Period Monopoly with an Experience Good

Consider the following model of a two-period monopolist selling an *experience good*. The inverse market demand in the first period is

$$p(q_1) = 10 - q_1$$

In the second period the demand is:

$$p(q_2) = 10 + \frac{q_1}{2} - q_2$$

The amount the monopolist sells in the second period depends on how much the monopolist sold in the first period. The amount the monopolist sold in the first period shifts out the second period demand curve (raises the vertical intercept). The monopolist's cost function is $c(q_1, q_2) = 0$.

(i) If the monopolist was unaware he was selling an experience good and simply maximized profits in each period, what would his total profits be?

MC here is zero so to maximize profits the monopolist will set $MR = 0$. The marginal revenue in period 1 is $MR_1 = 10 - 2q_2$. Setting this equal to zero yields $q_1 = 5$. This gives $p_1 = 5$.

Given $q_1 = 5$, in the second period demand becomes $p_2 = 12.5 - q_2$. Marginal revenue is period 2 is then $MR_2 = 12.5 - 2q_2$. setting $MR_2 = 0$ yields $12.5 - 2q_2 = 0$, so $q_2 = 6.25$. Then $p_2 = 6.25$. Thus total profits are $25 + \frac{625}{16} = 64.0625$.

(ii) If the monopolist knows the way the demand curves depend on both quantities, write down his profit function for both periods. This should be a function of q_1 and q_2 only.

The profit function is:

$$\begin{aligned}\pi(q_1, q_2) &= q_1(10 - q_1) + q_2(10 + \frac{q_1}{2} - q_2) \\ &= 10q_1 - q_1^2 + 10q_2 + \frac{q_1 q_2}{2} - q_2^2\end{aligned}$$

(iii) Solve for the optimal prices and quantities by maximizing the profit function you wrote down above. Remember that maximizing a function of two variables requires setting both partial derivatives equal to 0 and solving the system of equations. Does the monopolist offer a low introductory price?

The setting the partial derivatives with respect to q_1 and q_2 equal to zero:

$$\begin{aligned}\frac{\partial \pi}{\partial q_1} &= 10 - 2q_1 + \frac{q_2}{2} = 0 \\ \frac{\partial \pi}{\partial q_2} &= 10 + \frac{q_1}{2} - 2q_2 = 0\end{aligned}$$

If we solve these equations, we find that $p_1 = \frac{10}{3}$, $p_2 = \frac{20}{3}$, $q_1 = \frac{20}{3}$ and $q_2 = \frac{20}{3}$. Thus, the monopolist offers a lower introductory price in the first period, and then raises it considerably in the second period.

(iv) Compare the profits from parts (i) and (iii). Is the monopolist better off in (i) or (iii)?

Plugging in the prices and quantities from (iii) we find that profits are $\frac{200}{3}$. Thus, the monopolist enjoys larger profits by charging the lower price in the first period to increase demand.