Choice and Demand

Question 1 – Neutral Goods

Revisiting the preferences from Question 6 on the previous problem set, suppose that you enjoy good 1 but you are neutral about good 2 (your indifference curves are vertical lines).

(i) Find the demand functions $x_1(p_1, p_2, m)$ and $x_2(p_1, p_2, m)$.

With these preferences you will spend all your money on good 1 and none on good 2. Therefore you demand functions would be:

$$x_1(p_1, p_2, m) = \frac{m}{p_1}$$
$$x_2(p_1, p_2, m) = 0$$

(ii) Write down a utility function $u(x_1, x_2)$ that would be consistent with these preferences.

A utility function that would be consistent with these preferences is:

$$u(x_1, x_2) = x_1$$

There are many functions that would work here, as long as they satisfied the following properties:

- Utility increased as x_1 increased.
- Utility does not change as x_2 changes.

Question 2 – Cobb-Douglas Demand

Suppose you have the following utility function for goods 1 and 2:

$$u(x_1, x_2) = 2x_1^{\frac{1}{4}}x_2^{\frac{3}{4}}$$

(i) Find the demand functions $x_1(p_1, p_2, m)$ and $x_2(p_1, p_2, m)$.

At the optimum, $|MRS| = \frac{p_1}{p_2}$:

$$\frac{MU_1}{MU_2} = \frac{p_1}{p_2}$$
$$\frac{\frac{\partial u(x_1,x_2)}{\partial x_1}}{\frac{\partial u(x_1,x_2)}{\partial x_1}} = \frac{p_1}{p_2}$$
$$\frac{\frac{2}{4}x_1^{\frac{1}{4}-1}x_2^{\frac{3}{4}}}{\frac{6}{4}x_1^{\frac{1}{4}}x_2^{\frac{3}{4}-1}} = \frac{p_1}{p_2}$$
$$\frac{2x_1^{-1}}{6x_2^{-1}} = \frac{p_1}{p_2}$$
$$\frac{1}{3}\frac{x_2}{x_1} = \frac{p_1}{p_2}$$

So $x_2 = \frac{3p_1x_1}{p_2}$. Inserting this into the budget constraint:

$$\begin{aligned} p_1x_1 + p_2x_2 &= m \\ p_1x_1 + p_2\left(\frac{3p_1x_1}{p_2}\right) &= m \\ p_1x_1 + 3x_1p_1 &= m \\ 4x_1p_1 &= m \\ x_1 &= \frac{m}{4p_1} \end{aligned}$$

Then $x_2 = \frac{3p_1}{p_2}x_1 = \frac{3p_1}{p_2}\left(\frac{m}{4p_1}\right) = \frac{3}{4}\frac{m}{p_2}$. Therefore the demand functions are:

$$\begin{aligned} x_1\left(p_1, p_2, m\right) &= \frac{m}{4p_1} \\ x_2\left(p_1, p_2, m\right) &= \frac{3m}{4p_2} \end{aligned}$$

(ii) The price of good 1 is \$1 and the price of good two is \$3. You have \$60 to spend. How much will you consume of goods 1 and 2?

Using these numbers in the demand functions:

$$\begin{aligned} x_1 &= \frac{m}{4p_1} = \frac{60}{4 \times 1} = 15 \\ x_2 &= \frac{3m}{4p_2} = \frac{3 \times 60}{4 \times 3} = 15 \end{aligned}$$

(iii) How much of goods 1 and 2 will you buy if you had \$120 and the prices stayed the same? Are the goods normal or inferior?

Using these numbers in the demand functions:

$$\begin{aligned} x_1 &= \frac{m}{4p_1} = \frac{120}{4 \times 1} = 30 \\ x_2 &= \frac{3m}{4p_2} = \frac{3 \times 120}{4 \times 3} = 30 \end{aligned}$$

(iv) If the price of good 2 doubled (to \$6), what would happen to your demand for good 1?

Since p_2 doesn't enter into the demand function for good 1, nothing will happen to the demand for good 1.

(v) Which of the following statements is true and why?

- Good 1 is a substitute to good 2.

- Good 1 is a complement to good 2.
- Good 1 is neither a substitute nor a complement to good 2.

Since the demand for good 1 doens't change in response changes in the price of good 2, good 1 is neither a substitute nor a complement to good 2.

Question 3 – Income and Substitution Effects

Suppose your utility function is $u(x_1, x_2) = x_1^{\frac{2}{3}} x_2^{\frac{1}{3}}$. It can be shown that the resulting demand functions are $x_1(p_1, p_2, m) = \frac{2m}{3p_1}$ and $x_2(p_1, p_2, m) = \frac{m}{3p_2}$ (you should check this). Throughout this question, you have income of $m = 30$.

(i) If $p_1 = 1$ and $p_2 = 1$, how much of goods 1 and 2 would you buy? Sketch the budget line and optimal bundle on a graph.

You would buy $x_1 = \frac{2}{3\times 1} = 20$ of good 1 and $x_2 = \frac{30}{3\times 1} = 10$ of good 2. See the first graph below.

(ii) If the price of good 1 increases to $p_1 = 2$ and the price of good 2 remains at p_2, how much of goods 1 and 2 would you buy? Sketch the new budget line and optimal bundle on your graph.

You would buy $x_1 = \frac{2}{3\times 2} = 10$ of good 1 and $x_2 = \frac{30}{3\times 1} = 10$ of good 2. See the second graph below.

(iii) Suppose that after the price of good 1 increased, the government gave you just enough money so that you could afford the original bundle, i.e. the bundle in part(i). How much money would the government need to give you?

With the new prices, the slope of the budget line is -2. We know the budget line after the government handout needs to pass through the point $(20, 10)$ as that was your original bundle. Since a government handout is like a change in income, the budget line will have the same slope of -2. We know therefore that the budget line satisfies:

$$x_2 = a - 2x_1$$

where a is the vertical intercept. Since we know $(20, 10)$ is a point on this line, we can use that to solve for the vertical intercept:

$$10 = a - 2 \times 20 \implies a = 10 + 2 \times 20 = 10 + 40 = 50$$

The new budget line is therfore:

$$x_2 = 50 - 2x_1$$

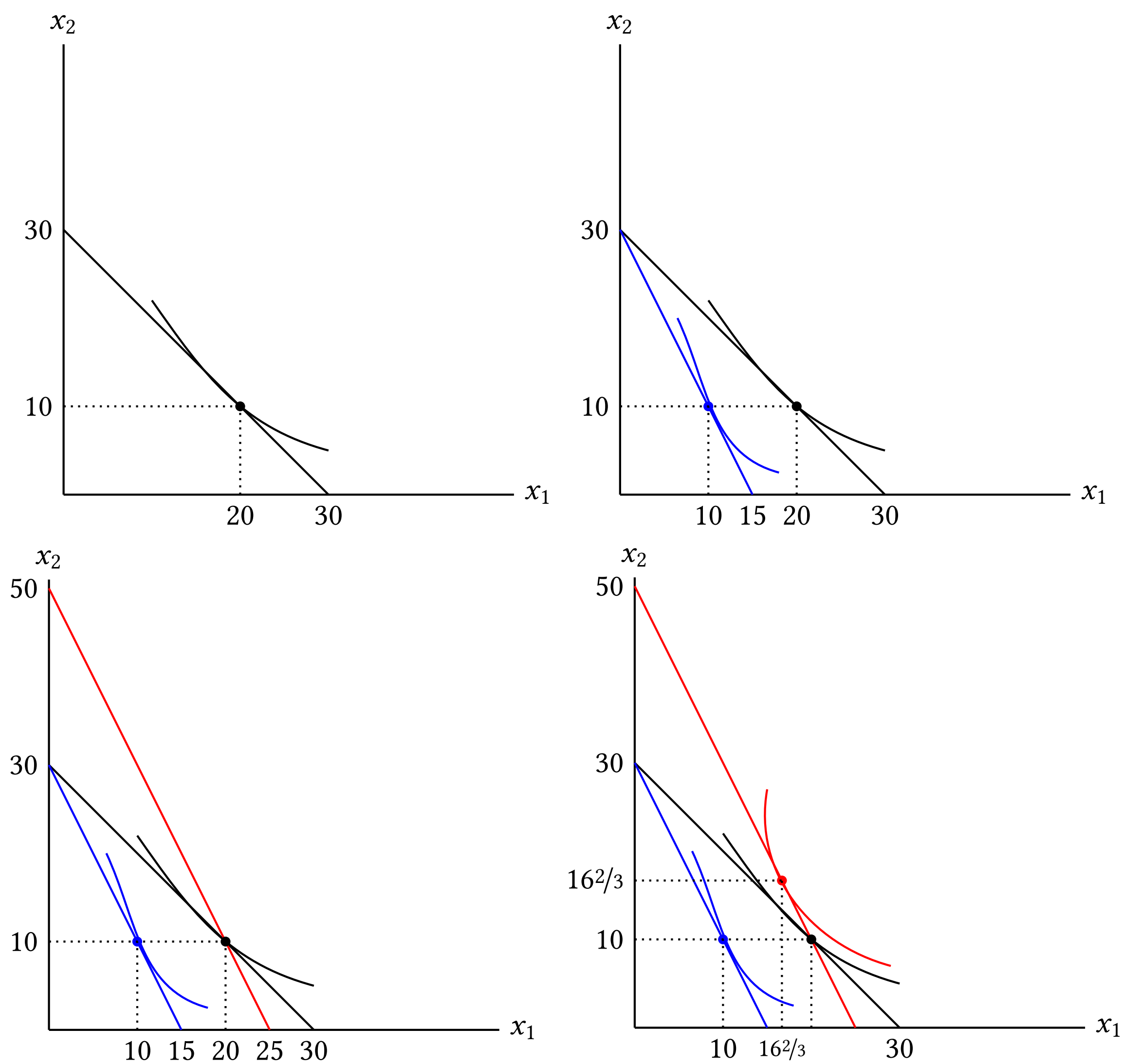
The vertical intercept is 50 and the horizontal intercept is 25. This is shown in the third graph below. Since $p_2 = 1$ and the vertical intercept is $\frac{m}{p_2}$, we know your income must be 50. Since your income was initially 30, the government would need to give you 20 to allow you to afford your initial bundle.

(iv) If the government gave you this reimbursement, how much of goods 1 and 2 would you buy? Is this different from part (i)? Sketch your new budget line after the government handout and the optimal bundle.

Going back to our demand functions with $m = 50$, we get $x_1 = \frac{2m}{3p_1} = \frac{2\times 50}{3\times 2} = \frac{100}{6} = 16\frac{2}{3}$ and $x_2 = \frac{m}{3p_2} = \frac{50}{3\times 1} = 16\frac{2}{3}$. Your new bundle is at $\left(16\frac{2}{3}, 16\frac{2}{3}\right)$. Before you bought $(20, 10)$ so your bundle does change as $\left(16\frac{2}{3}, 16\frac{2}{3}\right)$ gives higher utility (if you calculate the utilities you get $16\frac{2}{3}$ utility with the new bundle, and 15.87 with the old bundle).

(v) What is the income effect and substitution effect of the price increase of good 1 on the demand for good 1?

The substitution effect of the price change made your demand go from 20 to $16\frac{2}{3}$. The income effect is the remaining effect of going from $16\frac{2}{3}$ to 10. So the substitution effect is $3\frac{1}{3}$ ($\frac{1}{3}$ of the total effect) and the income effect is $6\frac{2}{3}$ ($\frac{2}{3}$ of the total effect).



Intertemporal Choice

Question 4 – Saving for Retirement

There are two time periods. In period 1 you work and earn income m_1. In period 2 you are retired and earn no income ($m_2 = 0$). The interest rate is r and there is no inflation. Your consumption in time periods 1 and 2 are c_1 and c_2 respectively.

The lifetime budget constraint is therefore:

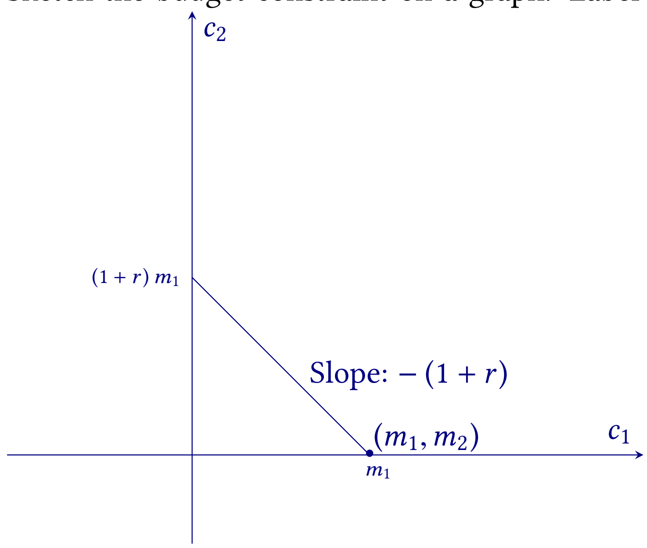
$$c_1 + \frac{c_2}{1+r} = m_1$$

Your lifetime utility function is:

$$u(c_1, c_2) = \frac{1}{2}c_1^2 + \frac{\beta}{2}c_2^2$$

where β is a constant.

(i) Sketch the budget constraint on a graph. Label the axis intercepts and the endowment.



(ii) The marginal rate of substitution of consumption between the two periods is:

$$MRS = -\frac{\frac{\partial u(c_1,c_2)}{\partial c_1}}{\frac{\partial u(c_1,c_2)}{\partial c_2}}$$

Calculate the MRS for the lifetime utility function $u(c_1, c_2)$ provided above.

$$\begin{aligned} MRS &= -\frac{\frac{\partial u(c_1,c_2)}{\partial c_1}}{\frac{\partial u(c_1,c_2)}{\partial c_2}} \\ &= -\frac{c_1}{\beta c_2} \end{aligned}$$

(iii) Now we want to find exactly how much you will want to save in the first period. In what follows, assume that $\beta = \frac{1}{1+r}$ (this will make the algebra easier).

At the optimum you will set the *MRS* equal to the slope of the budget constraint, which is $-(1+r)$. Therefore we have two equations to find c_1 and c_2:

$$\frac{\frac{\partial u(c_1,c_2)}{\partial c_1}}{\frac{\partial u(c_1,c_2)}{\partial c_2}} = 1 + r \tag{1}$$

$$c_1 + \frac{c_2}{1+r} = m_1 \tag{2}$$

Using these two equations, solve for c_1 and c_2 as a function of m_1 and r.

The first equation is:

$$\frac{c_1}{\beta c_2} = 1 + r$$

$$\frac{c_1}{c_2} = \beta(1+r)$$

Since $\beta = \frac{1}{1+r}$, we know that $\beta(1+r) = 1$. Then this becomes $c_1 = c_2$. Putting this in the budget constraint:

$$c_1 + \frac{c_2}{1+r} = m_1$$

$$c_1 + \frac{c_1}{1+r} = m_1$$

$$c_1\left(1 + \frac{1}{1+r}\right) = m_1$$

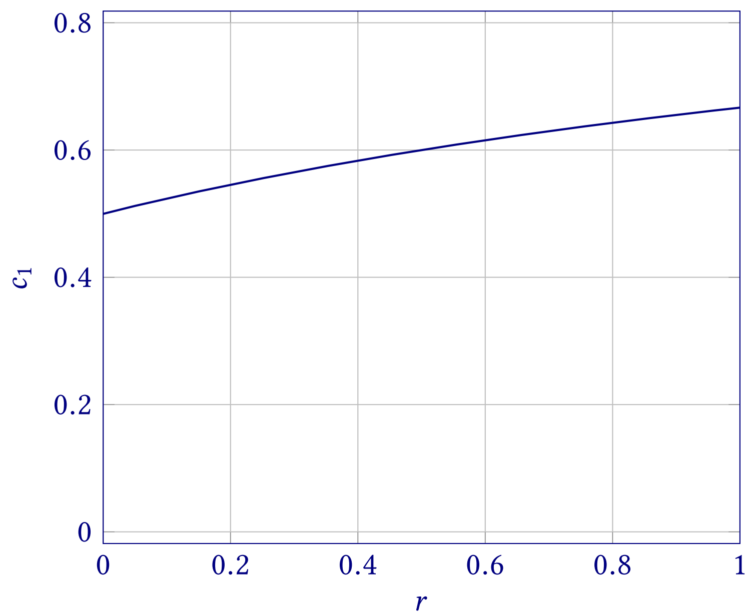
$$c_1\left(\frac{1+r}{1+r} + \frac{1}{1+r}\right) = m_1$$

$$c_1\left(\frac{2+r}{1+r}\right) = m_1$$

$$c_1 = m_1\left(\frac{1+r}{2+r}\right)$$

(iv) If the interest rate increases, will you consume more or less in the first period? To answer this you can assume that $m_1 = 1$ and r ranges from 0 to 1.

If we plot r against $\frac{1+r}{2+r}$, we get:



From this we can see that as r increases, the amount of consumption in the first period increases.

Uncertainty

Question 5 – Insurance

You have $200,000 worth of valuable items in your home. There is a 5% chance that you will be burgled. Fortunately, you keep $100,000 worth of your valuables in a hidden safe that the burgulars won't find. The local insurance company will insure your valuables for $7,000. That is, in the event of a burgulary the insurance company will give you $100,000 (the amount the burgulars stole).

Suppose your utility for wealth is $U(W) = \sqrt{W}$. So if you have $200,000 you utility is $\sqrt{200,000}$.

(i) What is your expected utility if you don't purchase insurance?

In the good state, you get $200,000. In the bad state you get $100,000. The expected utility is then:

$$0.05 \times \sqrt{100,000} + 0.95 \times \sqrt{200,000} = 0.05 \times 100 + 0.95 \times 141.42 = 440.6643$$

(ii) What is your expected utility if you do purhcase insurance?

The cost of insurance is $7,000. If you get burgled the insurance company will give you $100,000. Therefore in the good state *and* the bad state you get $\$200,000 - \$7,000 = \$193,000$. Your expected utility is then:

$$0.05 \times \sqrt{193,000} + 0.95 \times \sqrt{193,000} = \sqrt{193,000} = 439.3177$$

(iii) Would you purchase insurance?

Your expected utility when you don't purchase insurance is slightly higher than if you do purchase insurance. Therefore you will not purchase insurance. This is because the insurance is too expensive relative to the probability of the burgularly happening and the amount that would be stolen.

(iv) How much would the insurance policy be if it were actuarially fair?

If the insurance policy were actuarially fair, the cost of insurance should be the probability of the loss occuring times the value of the loss. Here that is $0.05 \times 100,000 = \$5,000$. The insurance company is charging $7,000, which is more than what is fair.

Technology

Question 6 – The Technical Rate of Substitutition (TRS)

Find the techinical rate of substitution for the following production function (show all your work, including the partial derivatives to get the marginal products):

$$f(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}$$

Interpret what the TRS means when $x_1 = 2$ and $x_2 = 4$.

$$MP_1 = \frac{\partial f(x_1, x_2)}{\partial x_1} = \frac{1}{3} x_1^{\frac{1}{3}-1} x_2^{\frac{2}{3}}$$

$$MP_2 = \frac{\partial f(x_1, x_2)}{\partial x_2} = \frac{2}{3} x_1^{\frac{1}{3}} x_2^{\frac{2}{3}-1}$$

$$TRS = -\frac{MP_1}{MP_2} = -\frac{\frac{1}{3} x_1^{\frac{1}{3}-1} x_2^{\frac{2}{3}}}{\frac{2}{3} x_1^{\frac{1}{3}} x_2^{\frac{2}{3}-1}} = -\frac{1}{2} \frac{x_1^{-1}}{x_2^{-1}} = -\frac{1}{2} \frac{x_2}{x_1}$$

When $x_1 = 2$ and $x_2 = 4$, the TRS becomes $-\frac{1}{2}\frac{4}{2} = -1$. If you reduce x_1 a small amount, you need an equal amount of x_2 extra to maintain the same level of output.

Question 7 – Returns to scale

Show whether the following production functions exhibit constant returns to scale, increasing returns to scale or decreasing returns to scale:

(i) $f(x_1, x_2) = x_1 x_2$.

$$f(tx_1, tx_2) = (tx_1)(tx_2) = t^2 x_1 x_2 = t^2 f(x_1, x_2)$$

This production function exhibits increasing returns to scale. For example, if $t = 2$ (you double you inputs), output will increase by $2^2 = 4$ times.

(ii) $f(x_1, x_2) = x_1 + x_2$

$$f(tx_1, tx_2) = tx_1 + tx_2 = t(x_1 + x_2) = tf(x_1, x_2)$$

This production function exhibits constant returns to scale. For example, if $t = 2$ (you double you inputs), output will also double.

(iii) $f(x_1, x_2) = \min\{x_1, x_2\}$.

Another way to write the min function is with cases:

$$\min\{x_1, x_2\} = \begin{cases} x_1 & \text{if } x_1 < x_2 \\ x_2 & \text{if } x_1 > x_1 \\ x_1 & \text{if } x_1 = x_1 \end{cases}$$

If $x_1 = x_2$, I could have written either x_1 or x_2 since they are the same (in fact, I could have replaced one of the strict inequalities with a weak inequality to replace the equality case). Now if we multiply x_1 and x_2 by t, where $t > 1$, then the inequalities don't change, i.e. if $x_1 < x_2$, then $tx_1 < tx_2$, and if $x_1 > x_2$, then $tx_1 > tx_2$. So:

$$\begin{aligned}\min\{tx_1, tx_2\} &= \begin{cases} tx_1 & \text{if } tx_1 < tx_2 \\ tx_2 & \text{if } tx_1 > tx_1 \\ tx_1 & \text{if } tx_1 = tx_1 \end{cases} \\ &= \begin{cases} tx_1 & \text{if } x_1 < x_2 \\ tx_2 & \text{if } x_1 > x_1 \\ tx_1 & \text{if } x_1 = x_1 \end{cases}\end{aligned}$$

The output of the function is scaled by t in each case, so:

$$f(tx_1, tx_2) = \min\{tx_1, tx_2\} = t\min\{x_1, x_2\} = tf(x_1, x_2)$$

This production function therefore exhibits constant returns to scale.

Profit Maximization

Question 8 – Short-run profit maximization

Suppose the production function is $y = f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$, the price of output is $p = 2$ and the prices of each input/factor are $w_1 = 1$ and $w_2 = 1$. The firm is competitive and therefore has no control over input and output prices. Factor 2 is fixed in the short run at $x_2 = 1$. Find the amount of factor 1 that will maximimize the firm's profits. What profit does the firm make?

The firm's profit is:

$$\pi = pf(x_1, x_2) - w_1 x_1 - w_2 x_2$$

Since $p = 2$, $w_1 = 1$, $w_2 = 1$ and $x_2 = \bar{x}_2 = 1$, the profit is:

$$\pi = 2x_1^{\frac{1}{2}} 1^{\frac{1}{2}} - x_1 - 1 \times 1$$

Simplifying:

$$\pi = 2x_1^{\frac{1}{2}} - x_1 - 1$$

With this, the firm's maximization problem is:

$$\max_{x_1} 2x_1^{\frac{1}{2}} - x_1 - 1$$

To maximize this, we take the derivative and set it equal to zero:

$$2 \times \frac{1}{2} \times x_1^{\frac{1}{2}-1} - 1 = 0$$

Solving for x_1:

$$x_1^{-\frac{1}{2}} = 1 \quad \Longrightarrow \quad x_1 = 1^{-2} = 1$$

The firm will optimally set $x_1 = 1$. With this we can calculate the firm's profits:

$$\pi = 2\,(1)^{\frac{1}{2}} - 1 - 1 = 0$$

With $x_1 = 1$, the firm manages to break even.