Mathematics Review

Question 1 – Differentiation

Differentiate the following functions, i.e. find $\frac{df(x)}{dx}$:

(i) $f(x) = 4x^2$

$$\frac{df(x)}{dx} = 8x$$

(ii) $f(x) = 201$

The derivative of a constant is zero:

$$\frac{df(x)}{dx} = 0$$

(iii) $f(x) = 2x + 4x^3$

$$\frac{df(x)}{dx} = 2 + 12x^2$$

Question 2 – Partial Differentiation

Find the partial derivatives of the following functions, i.e. find $\frac{\partial f(x_1,x_2)}{\partial x_1}$ and $\frac{\partial f(x_1,x_2)}{\partial x_2}$:

(i) $f(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}$

When we take the partial derivative of the function with respect to x_1, we treat x_2 as if it were a constant. Since the x_2 term is *multiplied* by the x_1 term, we just leave it there as it is:

$$\frac{\partial f(x_1, x_2)}{dx_1} = \frac{1}{3} x_1^{\frac{1}{3}-1} x_2^{\frac{2}{3}} = \frac{1}{3} x_1^{-\frac{2}{3}} x_2^{\frac{2}{3}}$$

For x_2 we follow the same approach:

$$\frac{\partial f(x_1, x_2)}{dx_2} = \frac{2}{3} x_1^{\frac{1}{3}} x_2^{\frac{2}{3}-1} = \frac{2}{3} x_1^{\frac{1}{3}} x_2^{-\frac{1}{3}}$$

(ii) $f(x_1, x_2) = 2x_1 + 3x_2^2$

When we take the partial derivative of the function with respect to x_1, we treat x_2 as a constant. Since the x_2 term is *not multiplied* by the x_1 term, it drops out when we take the partial derivative (because the derivative of just a constant is zero):

$$\frac{\partial f(x_1, x_2)}{dx_1} = 2$$

Similarly when we hold x_1 constant:

$$\frac{\partial f(x_1, x_2)}{dx_2} = 6x_2$$

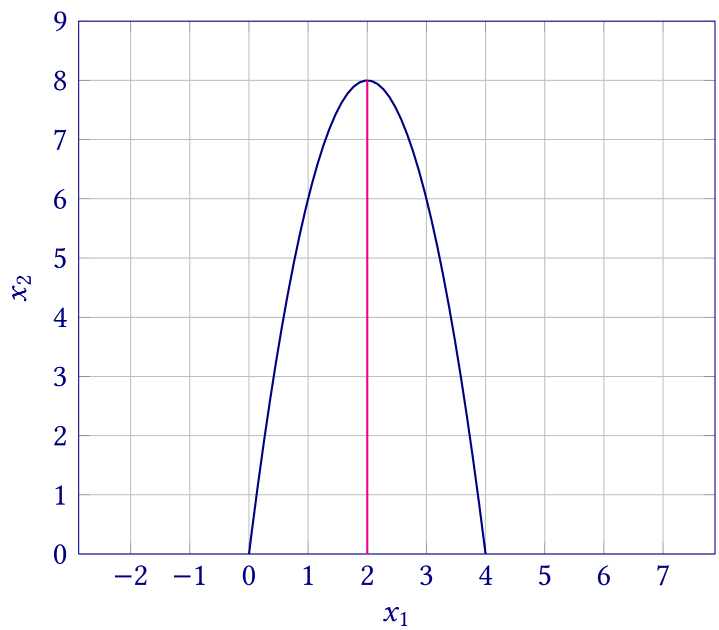
Question 3 – Optimization

For each of the functions below, do the following:

- Sketch the graph of the function.
- Find $\frac{df(x)}{dx}$
- Find the extreme point, i.e. at what $x^\star$ is $\frac{df(x^\star)}{dx} = 0$?
- Is the extreme point a maximum or a minimum? (To do this you will need to find the second derivative.)

(i) $f(x) = 8x - 2x^2$

Here is the function displayed as a graph:



The derivative is:

$$\frac{df(x)}{dx} = 8 - 4x$$

The extreme point happens when $\frac{df(x)}{dx} = 0$ So:

$$8 - 4x^\star = 0 \Rightarrow x^\star = \frac{8}{4} = 2$$

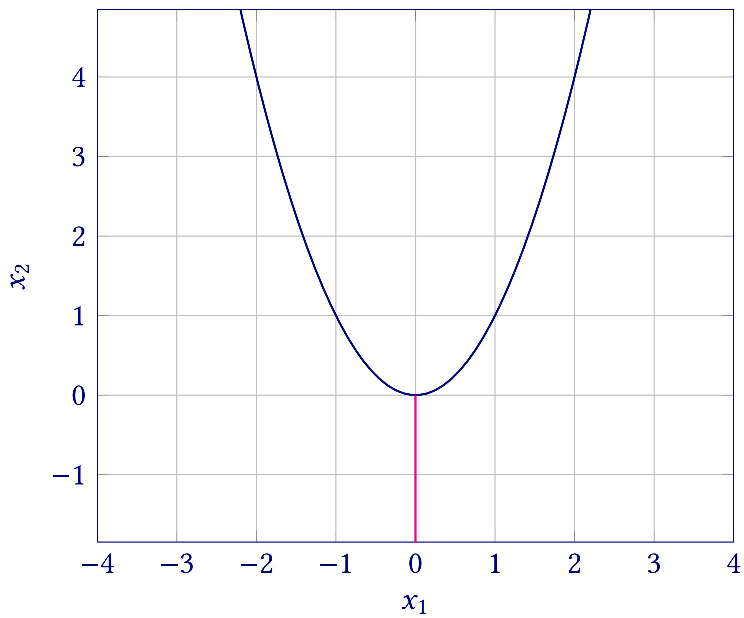
From the graph we can see that this is a maximum. However we can also use the second derivative to check if it is a maximum or a minimum. The second derivative is:

$$\frac{d^2 f(x)}{dx^2} = -4$$

When the second derivative is negative, we know that it is a maximum.

(ii) $f(x) = x^2$

Here is the function displayed as a graph:



The derivative is

$$\frac{df(x)}{dx} = 2x$$

$\frac{df(x)}{dx}$ is zero when $x = 0$ so $x^{\star} = 0$. From the graph we can see that this is a minimum. The second derivative is:

$$\frac{d^2 f(x)}{dx^2} = 2$$

Since this is positive, we know that it is a minimum.

The Budget Constraint

Question 4 – Two Price Changes

An individual's budget line is given by:

$$p_1 x_1 + p_2 x_2 = m$$

Suppose the price of good 1 doubled and the price of good 2 halved. Sketch the original and new budget constraints.

We saw in class that we can isolate x_2 on the left-hand side to write the budget constraint as

$$x_2 = \frac{m}{p_2} - \frac{p_1}{p_2} x_1$$

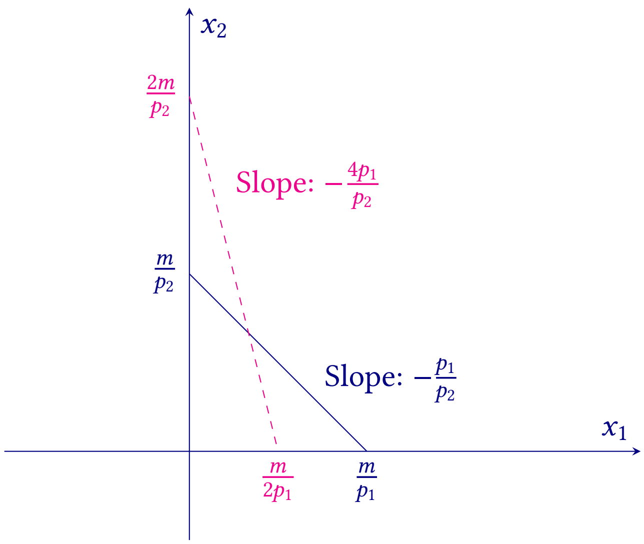
Thus the vertical intercept is at $\frac{m}{p_2}$ and the slope is $-\frac{p_1}{p_2}$. With the change in prices, the budget constraint becomes:

$$2p_1x_1 + \frac{1}{2}p_2x_2 = m$$

Rewriting this with x_2 isolated on the left-hand side is:

$$\frac{1}{2}p_2x_2 = m - 2p_1x_1$$
$$x_2 = \frac{2m}{p_2} - \frac{4p_1}{p_2}x_1$$

Thus the vertial intercept is now twice as high ($\frac{2m}{p_2}$ versus $\frac{m}{p_2}$) and the slope is four times as steep ($-\frac{4p_1}{p_2}$ versus $-\frac{p_1}{p_2}$). The old and new budget lines are graphed below.



The old budget line is the solid line and the new budget line is the dashed line.

Question 5 – Price and Income Changes

Suppose next year the prices of goods 1 and 2 will increase by 5% and your salary will also increase by 5%. Describe how this affects your budget line.

If all prices and income change by the same amount, the budget line will not change at all. To see this, start off with the original budget line:

$$p_1x_1 + p_2x_2 = m$$

Now let's increase prices and income by 5%. This is like multiplying prices and income by 1.05:

$$1.05p_1x_1 + 1.05p_2x_2 = 1.05m$$

If we divide both sides of the equation by 1.05, we end up with the original budget line:

$$p_1x_1 + p_2x_2 = m$$

Thus increasing all prices and income by the same amount does not affect the budget line.

Preferences

Question 6 – Indifference Curves

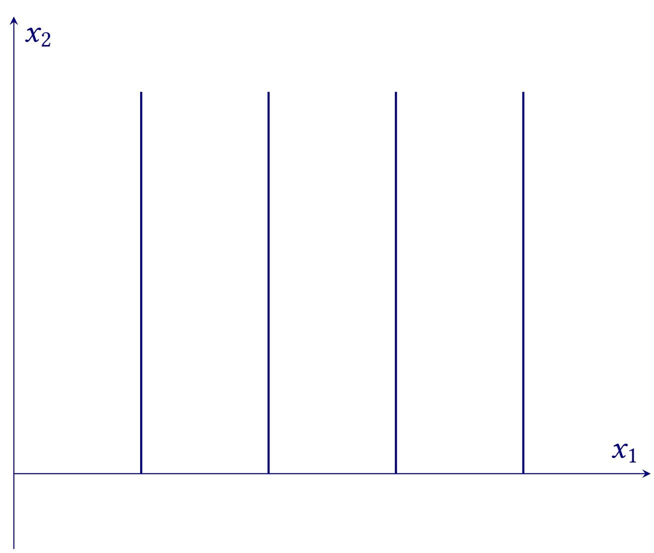
Suppose you enjoy consuming good 1 but you are not at all interested in good 2 (you are neutral about good 2). If you ever get any of good 2 you just give it away. What would your indifference curves look like? Sketch them.

Suppose we fix the amount of x_1 that you have (the good that you enjoy consuming). If we give you any more or less of x_2 (the good that you don't like), it will not change your utility. You are indifferent between any bundles that have more or less of x_2 as long as they have the same amount of x_1.

For example, suppose we started out with the bundle $(1, 1)$. You have one of good 1 and one of good 2. We want to try and find bundles that you would be indifferent to.

- $(1, 0)$ is one such bundle. You have the same amount of good 1 as before (the good that you like) but you no longer have any good 2. You are indifferent between $(1, 1)$ and $(1, 0)$ because you we were going to give away any x_2 that you had anyway, so if you don't have any it's the same as having 1 unit.
- $(1, 2)$ is another such bundle. You have the same amount of good 1 as before but you have extra good 2. Since you give away any good 2 it doesn't give you any extra utility. Hence you are indifferent.
- $(1, 10)$ is another bundle. It really doesn't matter how much x_2 you get as long as you have the same amount of x_1. You will always throw good 2 away.

The pattern here is that it doesn't matter how much x_2 you have as long as you have the same amount of good 1. Indifference curves are therefore vertical lines like this:



Utility

Question 7 – Indifference Curves and the Marginal Rate of Substitution

Answer the following questions about these utility functions:

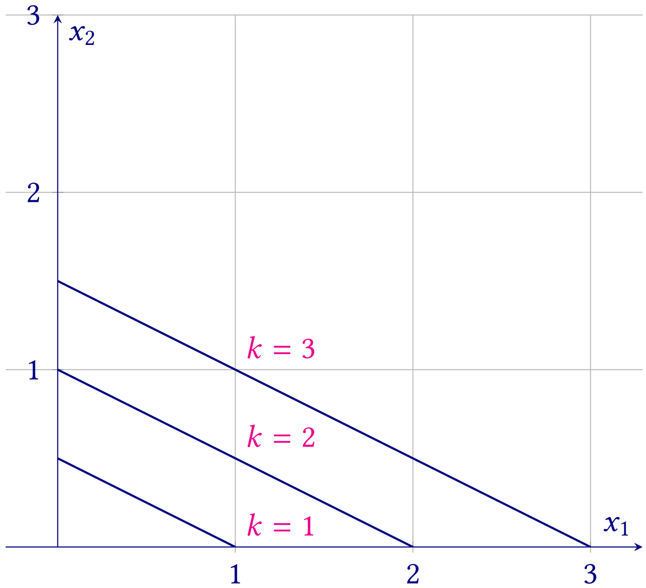
(i) $u(x_1, x_2) = x_1 + 2x_2$.

- Sketch the indifference curves of the utility function for fixed levels of utility $k = 1$, $k = 2$ and $k = 3$.

  We start with setting utility equal to k and then solving for x_2:

$$k = x_1 + 2x_2$$
$$x_2 = \frac{k}{2} - \frac{1}{2}x_1$$

  Now for $k = 1$, $k = 2$ and $k = 3$ we draw the lines:



- Calculate the marginal rate of substitution (MRS) between the two goods, where:

$$MRS = -\frac{\frac{\partial u(x_1,x_2)}{\partial x_1}}{\frac{\partial u(x_1,x_2)}{\partial x_2}}$$

  The marginal utility from x_1 is $\frac{\partial u(x_1,x_2)}{\partial x_1} = 1$ and the marginal utility from x_2 is $\frac{\partial u(x_1,x_2)}{\partial x_2} = 2$. The marginal rate of substitution is then:

$$MRS = -\frac{1}{2}$$

  This is precisely the slope of the indifference curves.

- Can you come up with an example of two goods where this utility function would be reasonable?

  An example of two "goods" that could satisfy these preferences are \$5 and \$10 bills. A \$10 bill would normally give people twice the utility as a \$5 bill (or you are indifferent between 2 \$5 bills and one \$10 bill). So x_1 could be a \$5 bill and x_2 could be a \$10 bill.

(ii) $u(x_1, x_2) = \min\left\{\frac{1}{2}x_1, x_2\right\}$

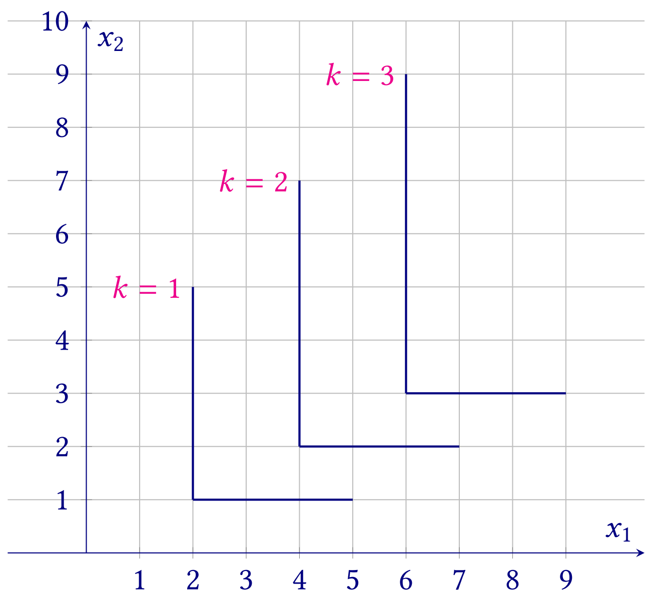
- Sketch the indifference curves of the utility function for fixed levels of utility $k = 1$, $k = 2$ and $k = 3$.

  If we set utility to k we get:

$$k = \min\left\{\frac{1}{2}x_1, x_2\right\}$$

  - If $x_2 \leq \frac{1}{2}x_1$, then this is the same as $k = x_2$, so solving for x_2 is just $x_2 = k$.
  - If $x_2 > \frac{1}{2}x_1$, then x_2 can be any number at all, provided it's at least $\frac{1}{2}x_1$.

  At the kink points we have $\frac{1}{2}x_1 = x_2$ so x_1 needs to be double x_2. Drawing the indifference curves gives:



- What is the marginal rate of substitution when $x_1 = 3$ and $x_2 = 1$? Interpret this number.

  When $x_1 = 3$ and $x_2 = 1$, we can see that the indifference curve is flat and hence its slope is 0. Since the slope of the indifference curve at a point is the same as the marginal rate of subsitution at that point, we have $MRS = 0$. This means that the consumer would be willing to give up a small amoumt of x_1 in return for zero x_2. The consumer is willing to throw away some x_1 without getting any x_2 in return.

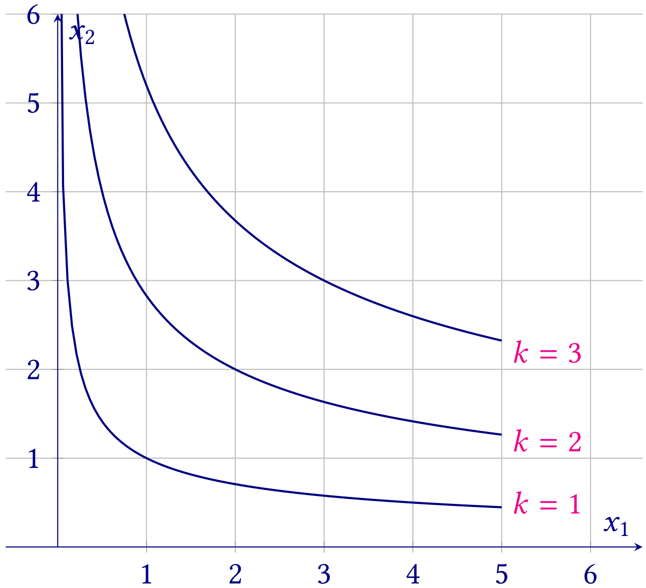
(iii) $u(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}$

- Sketch the indifference curves of the utility function for fixed levels of utility $k = 1$, $k = 2$ and $k = 3$.

  Setting utility equal to k and solving for x_2:

$$k = x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}$$
$$x_2^{\frac{2}{3}} = \frac{k}{x_1^{\frac{1}{3}}}$$
$$x_2 = \frac{k^{\frac{3}{2}}}{x_1^{\frac{1}{2}}}$$

  Graphing the indifference curves:



- Calculate the marginal rate of substitution between the two goods.

  The marginal utility from x_1 is $\frac{\partial u(x_1,x_2)}{\partial x_1} = \frac{1}{3}x_1^{\frac{1}{3}-1}x_2^{\frac{2}{3}}$ and the marginal utility from x_2 is $\frac{\partial u(x_1,x_2)}{\partial x_2} = \frac{2}{3}x_1^{\frac{1}{3}}x_2^{\frac{2}{3}-1}$. The marginal rate of substitution is then:

$$MRS = -\frac{\frac{1}{3}x_1^{\frac{1}{3}-1}x_2^{\frac{2}{3}}}{\frac{2}{3}x_1^{\frac{1}{3}}x_2^{\frac{2}{3}-1}} = -\frac{\frac{1}{3}x_1^{-1}}{\frac{2}{3}x_2^{-1}} = -\frac{x_2}{2x_1}$$